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EDITED AND PUBLISHED BY

ARTEMAS MARTIN, M. A.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

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THE
MATHEMATICAL VISITOR.

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Vol. 1. OCTOBER, 1878 No. 1.

INTRODUCTION.

In England and Europe periodical publications have contributed much to the diffusion of mathematical learning, and some of the greatest scientific characters of those countries commenced their mathematical career by solving the problems proposed in such works.

It was stated nearly three-quarters of a century ago that the learned Dr. Hutton declared that the *Ladies' Diary* had produced more mathematicians in England than all the mathematical authors in that kingdom.

Similar publications have produced like results in this country. Not a few of our ablest teachers and mathematicians were first inspired with a love of mathematical science by the problems and solutions published in the mathematical department of some unpretending periodical.

Turning over the leaves of the *Mathematical Miscellany*, published about forty years ago, we find there the names of Professors Peirce, Strong, Kirkwood, Root, Docharty, and many others, who have distinguished themselves in various departments of mathematics.

Some years ago we asked a young mathematician of great promise, who displayed great ingenuity and ability in solving difficult probability problems, what works he had treating on that subject; imagine our surprise when he replied that he had no works on probabilities, but had learned what he knew about that difficult branch of mathematical science by studying the problems and solutions in the *Schoolday Magazine.*

Believing that there is room for one more mathematical periodical, we have ventured to issue The MATHEMATICAL VISITOR, which will be devoted chiefly to problems and solutions; and we invite professors, teachers, students, and all lovers of the "bewitching science," to contribute their best problems and solutions for its pages.

Two lists of problems will be published in each number; one, headed "Junior Problems," for students and persons who have not advanced very far beyond the elementary branches; the other, headed "Senior Problems," will contain questions of a more difficult nature. For convenience of reference the problems will be numbered continuously.

A prize problem will be published in each number, for the best complete, correct, solution of which ten copies of The VISITOR will be given, and for the second best solution eight copies will be given.

All problems and solutions intended for publication in number two should be received by September 1, 1877.

Number two will be issued about the first of January, 1878; it will contain about 32 pages, and the price will be 50 cents.

Only a small edition will be printed, hence persons desiring to secure copies should send in their names at an early date.

ARTEMAS MARTIN, Lock Box 11, Erie, Pa.

NOTE.—The first edition being exhausted, a second edition is issued to supply the demand for this No. to complete sets. Some new matter has been inserted to fill out the last leaf. A. M.

JUNIOR PROBLEMS.

1.—Proposed by Prof. EDWARD BROOKS, M. A., Principal of Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

Two trains, one a and the other b feet long, move with uniform velocities on parallel rails; when they move in opposite directions they pass each other in m seconds, but when they move in the same direction the faster train passes the other in n seconds. Find the rate at which each train moves.

2.—Proposed by Mrs. ANNA T. SNYDER, Alquina, Fayette Co., Ind.

Two trees of equal hight stood on a level plane, 50 feet apart; one being cut off at the ground lodged against the other 20 feet from the top. What is the hight of the trees?

3.—Proposed by S. C. GOULD, Manchester, Hillsborough Co., N. H.

It is required to enclose the exact area of a given square by means of circular arcs only.

4.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London Co., Conn.

The straight line drawn, bisecting any angle of a triangle, to the opposite side, is less than half the sum of the sides containing that angle.

5.—Proposed by ARTEMAS MARTIN, Erie, Erie Co., Pa.

If 22 oxen and 28 cows eat 24 acres of grass in 18 weeks, and 20 oxen and 38 cows eat 30 acres of grass in 27 weeks, and 41 oxen and 26 cows eat 50 acres of grass in 60 weeks, how long will 40 acres of the same grass last 35 oxen and 14 cows, the grass in all cases growing uniformly?

6.—Proposed by THEO. L. DELAND, U. S. Treasury Department, Washington, D. C.

Four men, A, B, C and D, agreed to reap a circular field of grain in 28 days for \$249.36, and divide the money among them in proportion to the actual service each rendered. The first, second, third and fourth quadrants contained, respectively, wheat, oats, barley and rye. To the end of the 7th day A cut wheat, B oats, C barley and D rye; they each then advanced one quadrant, and to the end of the 13th day A cut oats, B barley, C rye and D wheat; they each again advanced one quadrant, and cut to the end of the 18th day; again they advanced one quadrant each and so worked to the end of the 22d day, and finished the wheat; again they advanced one quadrant each and finished the oats at the end of 3 days more; they again advanced one quadrant each and finished the barley at the end of the 27th day; and then advanced one quadrant each and completed the work according to contract.

Required—the time for each to reap the field, and an equitable division of the money, provided one quadrant is as difficult as another.

7.—Proposed by Prof. DAVID TROWBRIDGE, Waterburg, Tompkins Co., N. Y.

If A, B, C be the angles of a triangle, prove that

$$\frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}+\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}+\frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=1.$$

8.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In an equilateral triangle ABC lines drawn from each vertex to the opposite side, dividing that side so that the ratio of the side to one of the segments is n, form by their intersections another equilateral triangle; determine the ratio of the areas of these triangles.

9.—Proposed by Miss CHRISTINE LADD, Union Springs, Cayuga Co., N. Y.

Show that in any plane triangle

$$\frac{a\sin A+b\sin B+c\sin C}{a\cos A+b\cos B+c\cos C}=R\left(\frac{a^2+b^2+c^2}{abc}\right).$$

10.—Proposed by H. A. WOOD, M. A., Professor of Higher Mathematics, Keystone State Normal School, Kutztown, Berks Co., Pa.

A hollow paraboloid, depth 4 feet, stands on its vertex with axis vertical. After a shower the depth of water in the paraboloid, measured on the axis, was 2 inches. What was the uniform depth of the rainfall, the radius of the top of the paraboloid being three feet?

11.—Proposed by B. F. BURLESON, Oneida Castle, Oneida Co., N. Y.

The sides of a quadrilateral field are, respectively, and in order, 17, 35, 40 and 45 rods in length.

Determine the length of the straight line which, in passing through a point equally distant from all its angles, shall divide the field into two equal parts.

12.—Proposed by JAMES MCLAUGHLIN, Mantorville, Dodge Co., Minn.

A gentleman has a rectangular garden in latitude 41° 40′ north, the diagonal of which is a meridian line. At the south corner of the garden stands a perpendicular pine tree, the shade of which at noon, when the days and nights are equal, extends to a small rivulet which comes in at the east corner of the garden and runs due west through it, crossing the said meridian line 240 feet from the north corner of the garden, and going out of it 135 feet from the pine tree.

Required the area of the garden and the hight of the tree.

13.—Proposed by E. P. NORTON, Allen, Hillsdale Co., Mich.

In a triangle ABD the base AB is 40 rods. A line AC drawn from the angle A, and perpendicular to AB, intersecting BD in C, is 9 rods. If a point F be taken in AB, 10 rods from B, and a line drawn from this point through C, and produced, it will cut AD produced in a point E, 5 rods from D. Required the sides of the triangle.

14.—Proposed by Prof. HORATIO M. BLOOMFIELD, Reading, Berks Co., Pa.

Given, $w^2+y^2=c,\quad z^2+y^2=b,\quad z^2+x^2=a,\quad x+y=w+z;$ to find the value of x by quadratics.

15.—Proposed by HENRY HEATON, B. S., DesMoines, Polk Co., Iowa.

If n persons meet by chance, what is the probability that they all have the same birth-day, supposing every fourth year a leap year?

Solutions of these problems should be received by September 1, 1877.

SENIOR PROBLEMS.

16.—*Proposed by* Prof. David Trowbridge, Waterburg, Tompkins Co., N. Y.

Required the expansion of θ^2 in a series of cosines of multiples of θ, thus:

$$\theta^2 = A_0 + A_1\cos\theta + A_2\cos 2\theta + \ . \ . \ . \ + A_n\cos n\theta + \ . \ . \ .$$

17.—*Proposed by* Miss Christine Ladd, Union Springs, Cayuga Co., N. Y.

The polar with respect to a triangle of its orthogonal center is the radical axis of the circumscribing circle, the nine-point circle and the circle with respect to which the triangle is self-conjugate.

18.—*Proposed by* I. H. Turrell, Cumminsville, Hamilton Co., Ohio.

Suppose n fixed lines, radiating from a fixed point, are touched two and two by n circles of given radii; that is, that each line touches two of the circles, and each circle touches two of the lines. The order of arrangement being known, it is required to construct geometrically a polygon of n sides, whose vertices shall lie on the n lines, and whose sides shall touch the given circles.

19.—*Proposed by* Dr. S. H. Wright, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, Yates Co., N. Y.

In latitude $\lambda = 42°$N., when the sun's declination north $= \delta = 20°$, the sun shone on the top of a high peak in a mountain $\beta = 4$ minutes before it shone on the plane of the horizon. How high was the peak?

20. Proposed by William Hoover, Superintendent of Public Schools, Bellefontaine, Logan Co., O.

The weights W_1 and W_2 are suspended, θ degrees apart, from the circumference of a vertical circle free to move about its center. Find the angle made by the radii drawn from the points of suspension with the horizontal through the center of the circle when the system is in equilibrium.

21.—Proposed by Dr. David S. Hart, M. A., Stonington, New London Co., Conn.

Required five numbers whose sum shall be a biquadrate, and the sum of any four of them a square.

22. Proposed by Walter Siverly, Oil City, Venango Co., Pa.

An ellipsoid, axes $2a$, $2b$, $2c$, is intersected by a cylinder, radius b, the axis of the cylinder coinciding with the axis $2c$ of the ellipsoid. Find the volume common to both.

23. Proposed by Artemas Martin, Erie, Erie Co., Pa.

Find the least integral values of x and y that will satisfy the equation

$$x^2 - 9781y^2 = 1.$$

24.—Proposed by James McLaughlin, Mantorville, Dodge Co., Minn.

An army in the form of a circle, $2a$ miles in diameter, marches due north at the uniform rate of n miles an hour. An officer starts from the rear of the army and rides around it at the uniform rate of m miles an hour, keeping close to the army all the time. Required the equation of the *curve* the officer describes, and the distance he rides while going once around the army.

25. Proposed by DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson Co., N. J.

A perfectly homogeneous sphere has an angular velocity ω about its diameter. If the sphere gradually contract, remaining constantly homogeneous, required the angular velocity when it has half the original diameter.

26.—Proposed by E. B. SEITZ, Greenville, Darke Co., Ohio.

Two equal spheres, radii r, are described within a sphere, radius $2r$; find the average of the volume common to the two spheres.

27.—Proposed by ARTEMAS MARTIN, Erie, Erie Co., Pa.

The first of two casks contains a gallons of wine, and the second b gallons of water. Part of the water is poured into the first cask, and then part of the mixture is poured back into the second. Required the probability that not more than $\frac{1}{n}$ of the contents of the second cask is wine.

28.—Proposed by F. P. MATZ, B. E., B. S., Reading, Berks Co., Pa.

The center of a variable circular disk moves along the major axis, between the left-hand extremity and center, of a variable elliptic surface; to determine the average area of all variable ellipses whose major axes are coincident with that of the first elliptic surface and whose right-hand vertex is limited by the right-hand extremity of said elliptic surface while the left is tangent to the periphery of the disk.

29.—Proposed by E. B. SEITZ, Greenville, Darke Co., Ohio.

Two points are taken at random in the surface of a given circle, but on opposite sides of a given diameter. Find (1) the chance that the distance between the points is less than the radius of the circle, and (2) the average distance between them.

30.—Proposed by ARTEMAS MARTIN, Erie, Erie Co., Pa.

A straight tree growing vertically on the side of a mountain was broken by the wind, but not severed; find the chance that the top reaches to the ground.

31.—Proposed by ARTEMAS MARTIN, Erie, Erie Co., Pa.

A sphere, radius r, and a candle are placed at random on a round table, radius R, the hight of the candle being equal to the radius of the sphere. Find the average surface of the sphere illuminated by the candle.

32.—Proposed by E. B. SEITZ, Greenville, Darke Co., Ohio.

A radius is drawn dividing a given semicircle into two quadrants, and a point taken at random in each quadrant; find the average distance between them.

33.—Proposed by E. B. SEITZ, Greenville, Darke Co., Ohio.

Three points being taken at random within a sphere, find the average area of the triangle formed by joining them.

34.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, Erie, Erie Co., Pa.

A boy stepped upon a horizontal turn-table, while it was in motion, and walked across it, keeping all the time in the same vertical plane. The boy's velocity is supposed to be uniform in his track on the table, and the motion of the table towards him. The velocity of a point in the circumference of the turn-table is n times the velocity of the boy along the curve he describes.

Required—the nature of the *curve* the boy describes on the table, and the distance he walks while crossing it (1) when n is less than 1, (2) when $n=1$ and (3) when n is greater than 1.

Solutions of these problems should be received by September 1, 1877.

UNSOLVED PROBLEMS.

By Artemas Martin, Erie, Pa.

I am not aware that the following problems have ever been completely solved.

Perhaps some of those in the Diophantine Analysis may be impossible.

If correct solutions to any of these problems are received in time, they will be published in the next number of the Visitor.

1. To find three biquadrate numbers whose sum is a biquadrate number.

2. To find four biquadrate numbers whose sum is a biquadrate number.

3. To find four positive whole numbers, the sum of every two of which is a biquadrate.

4. To find three square numbers whose sum is a square, such that the sum of every two is also a square.

5. To find four square numbers such that the sum of every two of them shall be a square.

6. To find four positive cube numbers, the sum of any three of which is a cube.

7. To find three positive integral numbers the sum of whose squares is a square, and the sum of their cubes a cube.

8. To find three positive whole numbers whose sum, sum of squares, and sum of biquadrates are all rational squares.

9. To find three positive whole numbers whose sum, sum of squares, and sum of cubes shall all be rational cubes.

10. To find three positive integral numbers whose sum, sum of squares, and sum of biquadrates shall all be rational cubes.

11. To find five integral numbers, the sum of the squares of every four of which shall be a square.

12. A coin, radius r, is placed at the bottom of an empty hemispherical bowl, radius R, and the bowl is filled with water, when the whole coin is just visible to an eye in a given position looking over the edge. How much must the bowl be tipped *from* the eye to hide the coin?

13. Two horses, coupled together with a rope b feet long, run at uniform rates of speed on a circular race-course whose radius is a feet, the fleeter horse being outside of the track and the other *on* it. They start even, with the rope stretched. Required the *curve* described by the fleeter horse, and the distance each has run when he comes to the track.

14. A cylindrical cask, radius R inches and depth a inches, is full of wine. Through a pipe in the top, water can be let in at the rate of b gallons per minute, and through a pipe, radius r inches, in the center of the bottom the mixture can escape at a faster rate when the cask is full. If both pipes be opened at the same instant, how much wine will remain in the cask at the end of t minutes, supposing the two fluids to mingle perfectly?

15. From an urn containing 20 balls, a number of balls are taken at random and put into a bag, and then a number are taken at random from the bag and put into a box; if, now, a number of balls be taken at random from the box, what is the probability that the number is odd?

16. If n bricks be piled upon one another at random, (like they would be in a wall one brick thick,) what is the probability that the pile will not fall down?

17. Three radii are drawn at random in a circle and a point is taken at random in each sector; find the chance that the triangle formed by joining them is acute.

18. A round flat pie is cut into three equal pieces; if they be piled up at random on a level table, what is the probability that the pile will not fall down?

19. Three pennies are thrown horizontally at random, one at a time, into a circular box whose diameter is twice the diameter of a penny; find the chance that only one of the pennies rests on the bottom of the box.

20. Three points are taken at random, one in each of the sides of a given triangle; find the chance that the triangle formed by joining them is acute.

21. If four pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

22. Three points are taken at random in the surface of a given triangle; find the chance that the triangle formed by joining them is acute.

23. If four dice be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

24. A plank, length $2a$, is cut at random into three rectangular pieces; if the pieces be piled up at random on a horizontal plane, (like three bricks in a wall,) what is the probability that the pile will not fall down?

25. Three arrows are sticking in a circular target; what is the probability that the distance between any two of them does not exceed the radius of the target?

26. If five pennies be piled up at random on a horizontal plane, what is the chance that the pile will not fall down?

27. The bottom of a circular box is covered with an adhesive substance. Two rods equal in length to the radius of the box are dropped horizontally into it at random, one at a time. What is the chance that the rods are crossed in the box?

28. Find the average distance between two points taken at random in the surface of a given rectangle, one on each side of a diagonal.

29. A straight rod, length $2a$ and mass M, is suspended by a string attached to its middle point. A mouse, mass m, crawls down the string and along the rod to the end. Required (1) the equation to the curve the mouse describes in space, (2) the equation to the curve the rod always touches and (3) the inclination of the rod to the vertical when the mouse is at the end.

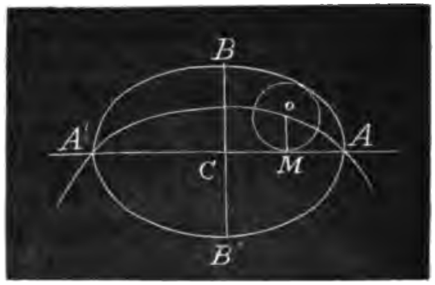
30. A straight rod, length $2a$ and mass M, is balanced on a cylinder, radius r. A large caterpillar, mass m, crawls from the middle of the rod towards one end. Required (1) the equation to the curve the caterpillar describes in space, and (2) the inclination of the rod to the horizon when the caterpillar is at the end, the surfaces of the rod and cylinder being rough enough to prevent sliding.

SOLUTION OF A PROBLEM.

By E. B. Seitz, Greenville, Ohio.

Find the equation to the locus of the centers of all the circles that can be inscribed in a given semi-ellipse

[This is problem 87 proposed in *Our Schoolday Visitor* for June, 1871, p. 164. No solution was published.—Ed.]



Put CM$=x$, OM$=y$; then representing the co-ordinates of the point of tangency of the circle and ellipse by m, n, we have for the equation to the circle

$$(n-y)^2+(m-x)^2=y^2 \quad \ldots\ldots (1),$$

and for the equation to the ellipse

$$a^2n^2+b^2m^2=a^2b^2 \quad \ldots\ldots\ldots (2),$$

Differentiating (1) and (2), we have

$$\frac{dm}{dn}=-\frac{n-y}{m-x}, \text{ and } \frac{dm}{dn}=-\frac{a^2n}{b^2m}.$$

But for the point of tangency of the circle and ellipse, these two values of $\frac{dm}{dn}$ are equal;

$$\therefore \quad \frac{n-y}{m-x}=\frac{a^2n}{b^2m} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (3) $n=\frac{b^2my}{a^2x-(a^2-b^2)m}$. Substituting this value of n in (2), clearing of fractions and reducing, we have

$$(a^2-b^2)^2m^4-2(a^2-b^2)a^2xm^3+[a^2x^2+b^2y^2-(a^2-b^2)^2]a^2m^2+2(a^2-b^2)a^4xm-a^6x^2=0 \quad (4).$$

From (3) and (2) $\quad \left(\frac{n-y}{m-x}\right)^2=\frac{a^4n^2}{b^4m^2}=\frac{a^2(a^2-m^2)}{b^2m^2},$

and from (1) $\quad \left(\frac{n-y}{m-x}\right)^2=\left(\frac{y}{m-x}\right)^2-1;$

$$\therefore \quad \frac{a^2(a^2-m^2)}{b^2m^2}=\left(\frac{y}{m-x}\right)^2-1,$$

or $\quad (a^2-b^2)m^4-2(a^2-b^2)xm^3+[(a^2-b^2)x^2+b^2y^2-a^4]m^2+2a^4xm-a^4x^2=0 \quad \ldots (5).$

Subtracting (4) from (a^2-b^2) times (5), we have

$$2(a^2-b^2)xm^3-[(2a^2-b^2)x^2+b^2y^2+a^2(a^2-b^2)]m^2+a^4x^2=0 \quad \ldots\ldots (6).$$

Subtracting (4) from a^2 times (5), we have

$$(a^2-b^2)m^3-(x^2+2a^2-b^2)a^2m+2a^4x=0 \quad \ldots\ldots\ldots\ldots (7).$$

Subtracting (6) from $2x$ times (7), we have

$$[(2a^2-b^2)x^2+b^2y^2+a^2(a^2-b^2)]m^2-2(x^2+2a^2-b^2)a^2xm+3a^4x^2=0 \quad \ldots (8).$$

Subtracting x times (7) from twice (6), we have

$$3(a^2-b^2)xm^2-2[(2a^2-b^2)x^2+b^2y^2+a^2(a^2-b^2)]m+a^2x(x^2+2a^2-b^2)=0 \quad \ldots (9).$$

Subtracting $[(2a^2-b^2)x^2+b^2y^2+a^2(a^2-b^2)]$ times (9) from $3x(a^2-b^2)$ times (8), we have

$$2[(a^4-a^2b^2+b^4)x^4+2(2a^2-b^2)b^2x^2y^2+b^4y^4-(a^2-b^2)(2a^2-b^2)a^2x^2+2(a^2-b^2)a^2b^2y^2+a^4(a^2-b^2)^2]m$$
$$-a^2x[(2a^2-b^2)x^4+b^2x^2y^2-(4a^4-4a^2b^2-b_4)x^2+(2a^2-b^2)b^2y^2+a^2(a^2-b^2)(2a^2-b^2)]=0 \quad . (10).$$

Subtracting $3a^2x$ times (9) from $(x^2+2a^2-b^2)$ times (8), we have

$$[(2a^2-b^2)x^4+b^2x^2y^2-(4a^4-4a^2b^2-b^4)x^2+(2a^2-b^2)b^2y^2+a^2(a^2-b^2)(2a^2-b^2)]m$$
$$-2a^2x[x^4-(2a^2-b^2)x^2-3b^2y^2+a^4-a^2b^2+b^4]=0 \quad \ldots\ldots\ldots \quad (11).$$

Eliminating m from (10) and (11), we have

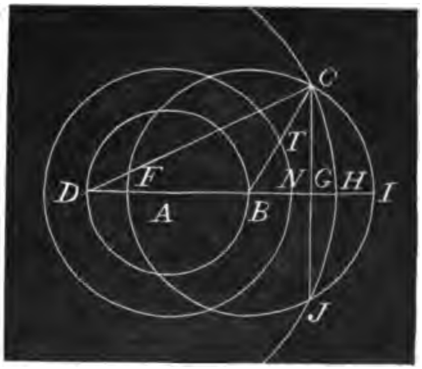
$$b^2x^8+2(2a^2-b^2)x^6y^2+b^2x^4y^4-2(2a^2-b^2)b^2x^6-4(3a^4-3a^2b^2+2b^4)x^4y^2$$
$$-10(2a^2-b^2)b^2x^2y^4-4b^4y^6+(6a^4-6a^2b^2+b^4)b^2x^4+2(2a^2-b^2)(3a^4-3a^2b^2+b^4)x^2y^2$$
$$-(8a^4-8a^2b^2-b^4)b^2y^4-2(a^2-b^2)(2a^2-b^2)a^2b^2x^2-2(a^2-b^2)(2a^4-2a^2b^2-b^4)a^2y^2+a^4b^2(a^2-b^2)^2=0,$$

which is the required equation.

SOLUTION OF A PROBLEM IN PROBABILITIES.

By Artemas Martin.

If three pennies be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?



The pile will stand if the common center of gravity of the second and third coins falls on the surface of the first or bottom coin.

Let r be the radius of a penny; then the center of the second coin may fall anywhere in a circle whose radius is $2r$ and center the center of the surface of the first or bottom coin, and the center of the third coin may fall anywhere in a circle whose radius is $2r$ and the center the center of the surface of the second coin.

The number of positions of the center of the second coin is therefore proportional to $4\pi r^2$, and for every one of these positions the center of the third coin can have $4\pi r^2$; hence the total number of positions of the second and third coins is proportional to $16\pi^2r^4$.

We must now determine in how many of these $16\pi^2r^4$ positions the pile will stand.

Let A be the center of the first or bottom coin, and B the center of the second coin. Take AD$=$AB, and with center D and radius $2r$ describe the arc CHJ. If the center of the third coin is on the surface CFJH, the second and third coins will remain on the first, since BN$=$NH, BT$=$TC, and the pile will not fall down.

When AB is not greater than $\frac{1}{2}r$, the circle CHJ will not cut the surface of the second coin, and the pile will stand if the center of the third coin is anywhere on the second.

Let AB$=$AD$=x$, $S=$surface CFJH, and $p=$the probability required;

then $\quad \text{DB}=2x, \quad \text{BG}=\dfrac{3r^2-4x^2}{4x}, \quad \text{DG}=\dfrac{3r^2+4x^2}{4x},$

$$\text{CG}=\frac{1}{4x}\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}}, \text{ arc CI}=r\cos^{-1}\Big(\frac{3r^2-4x^2}{4rx}\Big), \text{ arc CH}=2r\cos^{-1}\Big(\frac{3r^2+4x^2}{8rx}\Big)$$

$$S=\pi r^2+4r^2\cos^{-1}\Big(\frac{3r^2+4x^2}{8rx}\Big)-r^2\cos^{-1}\Big(\frac{3r^2-4x^2}{4rx}\Big)-\tfrac{1}{2}\Big[16r^4-(5r^2-4x^2)^2\Big]^{\frac{1}{2}},$$

and $$p=\frac{1}{16\pi^2r^4}\int_0^{\frac{1}{2}r}\pi r^2.2\pi x dx+\frac{1}{16\pi^2r^4}\int_{\frac{1}{2}r}^{r}S.2\pi x dx,=\frac{1}{16}+\frac{1}{8\pi r^4}\int_{\frac{1}{2}r}^{r}(S-\pi r^2)x dx.$$

$$\int r^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)xdx=\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$
$$+\frac{1}{16}r^2\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}},$$

$$\int 4r^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)xdx=2r^2x^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)+\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)$$
$$-\tfrac{1}{4}r^2\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}},$$

$$\int\tfrac{1}{2}\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}xdx=\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)-\frac{1}{32}(5r^2-4x^2)\left[16r^4-(5r^2-4x^2)^2\right]^{\frac{1}{2}}.$$

$$\therefore\quad p=\frac{1}{16}-\frac{1}{8\pi r^4}\Bigg[\tfrac{1}{2}r^2x^2\cos^{-1}\left(\frac{3r^2-4x^2}{4rx}\right)-2r^2x^2\cos^{-1}\left(\frac{3r^2+4x^2}{8rx}\right)$$
$$-\tfrac{1}{2}r^4\cos^{-1}\left(\frac{5r^2-4x^2}{4r^2}\right)+\frac{1}{32}(5r^2-4x^2)\sqrt{16r^4-(5r^2-4x^2)^2}\Bigg]_{\frac{1}{2}r}^{r},$$

$$=\frac{1}{16}-\frac{3}{16\pi}\left(\frac{3}{16}\sqrt{15}-2\sin^{-1}\frac{1}{4}\right).$$

THE INTRINSIC EQUATION OF A CURVE.

By Artemas Martin, M. A., Member of the London Mathematical Society.

I am not aware that this subject has been treated in an American work.

If s denote the length of an arc of a curve measured from a fixed point, φ the inclination of the tangent at the variable extremity of the arc to the tangent at the fixed point, then $s=f(\varphi)$, the relation between s and φ, is called the *intrinsic equation* of the curve.

Let $y=f(x)$ be the rectangular equation of a curve, the origin being a point on the curve, and the axis of y a tangent at that point; then $\frac{dy}{dx}=f'(x)=\cot\varphi$, from which x is known in the trigonometrical functions of φ, say $=F(\varphi)$; then $\frac{dx}{d\varphi}=F'(\varphi)$; also $\frac{ds}{dx}=\operatorname{cosec}\varphi$; therefore $\frac{ds}{d\varphi}=\operatorname{cosec}\varphi F'(\varphi)$; from this equation s may be found in terms of φ by integration. A similar result will be obtained when the axis of x is a tangent at the origin. If the given equation of the curve be $x=f(y)$, we may proceed in a similar manner.

Examples.—1. Required the intrinsic equation of the circle.

SOLUTION.

The required equation is obviously $s=a\varphi$, but we will deduce it from the ordinary equation by the method explained above.

The rectangular equation being $y=\sqrt{(2ax-x^2)}$, we have $\frac{dy}{dx}=\frac{a-x}{\sqrt{(2ax-x^2)}}=\cot\varphi$; whence $x=a(1\pm\cos\varphi)$, $\therefore\ \frac{dx}{d\varphi}=\pm a\sin\varphi$. Taking the lower sign, $\frac{ds}{d\varphi}=a$, and $s=a\varphi$.

2.—Required the intrinsic equation of the tractrix.

SOLUTION.

The retangular equation of the tractrix is

$x=a\log\left(\frac{a+\sqrt{(2ay-y^2)}}{a-y}\right)-\sqrt{(2ay-y^2)}$, the axis of y being a tangent at the origin.

$\therefore \frac{dy}{dx}=\frac{a-y}{\sqrt{(2ay-y^2)}}=\cot\varphi$; whence $y=a(1\pm\cos\varphi)$, $\therefore \frac{dy}{d\varphi}=a\sin\varphi$. But $\frac{ds}{dy}=\sec\varphi$,

$\therefore \frac{ds}{d\varphi}=a\tan\varphi$. Integrating, $s=a\log\sec\varphi+C$. When $s=0$, $\varphi=0$ and $C=0$;

$\therefore s=a\log\sec\varphi$ is the intrinsic equation of the tractrix.

3.—Required the intrinsic equation of the parabola.

SOLUTION.

From the equation $y^2=4ax$ we have $\frac{dy}{dx}=\frac{\sqrt{a}}{\sqrt{x}}=\cot\varphi$, $x=a\tan^2\varphi$,

$$\frac{dx}{d\varphi}=2a\tan\varphi\sec^2\varphi; \quad \therefore \quad \frac{ds}{d\varphi}=\frac{2a}{\cos^3\varphi},$$

whence by integration $s=\frac{a}{2}\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+\frac{a\sin\varphi}{1-\sin^2\varphi}$.

The intrinsic equation of a curve being given, to deduce the ordinary equation, we have, when the axis of y is a tangent at the origin, $\frac{dx}{ds}=\sin\varphi$, $\frac{dy}{ds}=\cos\varphi$;

$$\therefore \quad x=\int\sin\varphi ds, \qquad y=\int\cos\varphi ds.$$

When the axis of x is a tangent at the origin, $x=\int\cos\varphi ds$, $y=\int\sin\varphi ds$.

Now s being known in terms of φ from the intrinsic equation, by integration we can find x and y in terms of φ, and then by eliminating φ we obtain the ordinary equation of the curve in terms of x and y.

Example.—Required the rectangular equation of the curve whose intrinsic equation is $s=a\tan\varphi$.

SOLUTION.

$ds=a\sec^2\varphi d\varphi$, $\quad y=\int\frac{a\sin\varphi d\varphi}{\cos^2\varphi}=a\sec\varphi+C$,

$$x=\int\frac{a d\varphi}{\cos\varphi}=\tfrac{1}{2}a\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)+C'.$$

When $\varphi=0$, $x=0$, $y=0$; $\therefore$ $C=a$, $C'=0$, and

$$x=\tfrac{1}{2}a\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \;\ldots\; (1), \qquad y+a=a\sec\varphi \;\ldots\; (2).$$

But from (1) $\sec\varphi=\tfrac{1}{2}\left[\log^{-1}\left(\frac{x}{a}\right)+\log^{-1}\left(-\frac{x}{a}\right)\right]$, where the notation $\log^{-1} z$ has the same signification as e^z, e being the base of the Naperian system of logarithms; $\quad \therefore \quad y+a=\tfrac{1}{2}a\left[\log^{-1}\left(\frac{x}{a}\right)+\log^{-1}\left(-\frac{x}{a}\right)\right]$,

which is the well-known equation of the catenary.

NOTICES OF PERIODICALS.

THE EDUCATIONAL TIMES, AND JOURNAL OF THE COLLEGE OF PRECEPTORS.
MATHEMATICAL QUESTIONS, WITH THEIR SOLUTIONS, REPRINTED FROM THE EDUCATIONAL TIMES.
London, England: C. F. Hodgson & Son.

The *Educational Times* is published monthly, and contains, besides other valuable matter of a scholastic nature, an important Mathematical Department devoted to problems and solutions and short mathematical papers, edited by W. J. C. Miller, B. A., Registrar of the General Medical Council, to which many of the leading mathematicians of England, Europe and this country contribute.

Each number contains three or four large double column pages of mathematics. From 20 to 25 problems are proposed each month, and about half as many solutions are given.

Every mathematician who wishes to keep "posted" should take the *Educational Times*.

The *Reprint* is issued in half-yearly volumes of 112 pp, 8vo., boards, and contains, in addition to the mathematics published in the *Times*, abont as much more original matter. Twenty-five volumes have been published. Vol. XXVI will be ready soon. Vol. XXV contains solutions of 102 problems. These *Reprints* are rich in "Average," "Probability" and "Diophantine" solutions. In fact, nowhere else can be found so many elegant solutions of difficult problems relating to these fascinating branches of mathematical science. They are a mine of mathematical treasures; no mathematical library is complete without them.

The editor of the VISITOR will be happy to procure for American subscribers the *Times* at $2 a year, and the *Reprint* at $1.85 per vol.

THE ANALYST. A JOURNAL OF PURE AND APPLIED MATHEMATICS. Des Moines, Iowa: Edited and published by J. E. Hendricks, M. A.

The *Analyst* has entered upon its fourth volume, a feat never before accomplished by an American periodical devoted exclusively to mathematics. It is very efficiently and satisfactorily conducted, and deserves a large circulation.

The *Analyst* is issued in bi-monthly numbers of 32 pages at $2 a year. Each number contains several excellent papers and a well selected collection of problems and solutions.

EDUCATIONAL NOTES AND QUERIES: A MEDIUM OF INTERCOMMUNICATION FOR TEACHERS. Issued monthly, except in the vacation months, July and August. 8vo., 16 pp. Prof. W. D. Henkle, Editor, Salem, Ohio. Price, $1 per year.

Educational Notes and Queries contains, besides a vast fund of other instructive matter, a Mathematical Department devoted to problems, solutions and mathematical notes.

THE YATES COUNTY CHRONICLE, a weekly newspaper published at Penn Yan, N. Y., by the Chronicle Publishing Co., has a valuable Mathematical Department edited by Dr. S. H. Wright, M. A., Ph. D., occupying about a column, devoted to problems and solutions. $2 a year.

THE WITTENBERGER, a monthly college paper published at Springfield, Ohio, contains an excellent Mathematical Department admirably conducted by William Hoover, of Bellefontaine, Ohio. $1 a year.

THE NATIONAL EDUCATOR, published semi-monthly at Kutztown, Pa., by Uhrich & Gehring, has an interesting department of Science and Mathematics ably conducted by H. A. Wood, M. A., Professor of Higher Mathematics in the Keystone State Normal School. $1.25 a year.

THE MAINE FARMERS' ALMANAC, Masters & Livermore, Hallowell, Me., contains, besides the calendar and some interesting miscellaneous matter, a Puzzle Department and a Mathematical Department. The number for 1877 contains solutions to the 6 questions proposed last year and 7 new ones to be answered in the next number. Price 10 cents.

Vol. I. JANUARY, 1878. No. 2.

THE MATHEMATICAL VISITOR.

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THE MATHEMATICAL VISITOR.

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Vol. 1. JANUARY, 1878. No. 2.

SOLUTIONS OF JUNIOR PROBLEMS,

Proposed in No. 1.

1.—Proposed by Prof. Edward Brooks, M. A., Ph. D., Principal of Pennsylvania State Normal School, Millersville, Lancaster County, Pennsylvania.

Two trains, one a and the other b feet long, move with uniform velocities on parallel rails; when they move in opposite directions they pass each other in m seconds, but when they move in the same direction the faster train passes the other in n seconds. Find the rate at which the trains are moving.

I.—Solution by Dr. S. F. Bachelder, South Boston, Massachusetts; K. S. Putnam, Rome, Oneida County, New York; and W. L. Harvey, Maxfield, Penobscot Co., Maine.

Let x=rate per second, in feet, of the faster train, and y=rate of slower train.

When they move in opposite directions the rear ends of the trains approach each other at the rate of $x+y$ feet per second,

$$\therefore \frac{a+b}{x+y}=m \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1);$$

and when they move in the same direction the faster train gains on the slower at the rate of $x-y$ feet per second,

$$\therefore \frac{a+b}{x-y}=n \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (2).$$

Solving (1) and (2) we find

$$x=\frac{(a+b)\ (m+n)}{2mn}, \qquad y=\frac{(a+b)(n-m)}{2mn}.$$

Similar solutions furnished by *B. F. Burleson*, *William Hoover* and *O. D. Oathout*.

II.—Solution by Marcus Baker, U. S. Coast Survey Office, Washington, D. C.

Let v =the velocity at which the train a moves,
and v'= " " " " " b moves;
and suppose v greater than v'.

First; when the trains move in *opposite* directions.

Let x=the distance the train a moves, and y=the distance the train b moves in order to pass each other;

then $$x=mv \text{ and } y=mv',$$

whence $$x+y=a+b=m(v+v') \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (1).$$

Second; when the trains move in the *same* direction.

Let x'=the distance the train a moves, and y'=the distance the train b moves in order to pass each other;

then $x'=nv$ and $y'=nv'$,

whence $x'-y'=a+b=n(v-v')$ . . . . (2).

Dividing (1) by (2),

$$\frac{v+v'}{v-v'}=\frac{n}{m};$$

whence $$\frac{2v}{v+v'}=\frac{n+m}{n} \text{ and } \frac{2v'}{v-v'}=\frac{n-m}{n}.$$

Substituting for $v+v'$ and $v-v'$ their values from (1) and (2) we have

$$v=\frac{n+m}{2mn}(a+b), \qquad v'=\frac{n-m}{2mn}(a+b).$$

2.—Proposed by Mrs. Anna T. Snyder, Orange, Fayette County, Indiana.

Two trees of equal hight stood on a level plane, 50 feet apart; one being cut off at the ground lodged against the other 20 feet from the top. What is the hight of the trees.

I.—Solution by F. P. Matz, B. E., B. S., Reading, Berks County, Pennsylvania; and O. D. Oathout, Read, Clayton County, Iowa.

Put $50=m$, $20=n$, and the required hight h; then $m^2+(h-n)^2=h^2$.

Whence $$h=\frac{m_2+n^2}{2n}=72\tfrac{1}{2} \text{ feet.}$$

Solved in a similar manner by *B. F. Burleson* and *James McLaughlin.*

II.—Solution by D. W. K. Martin, Webster, Darke County, Ohio; and the Proposer.

Let $x=$ hight of the trees. Then after one of the trees is cut off and lodged we have a right-angled triangle, the base of which is 50 feet, or the distance between the trees; the perpendicular $x-20$ feet, or the distance from the plane to the top of the lodged tree, and the hypothenuse x, or the length of the tree cut off.

Consequently $$(50)^2+(x-20)^2=x^2$$

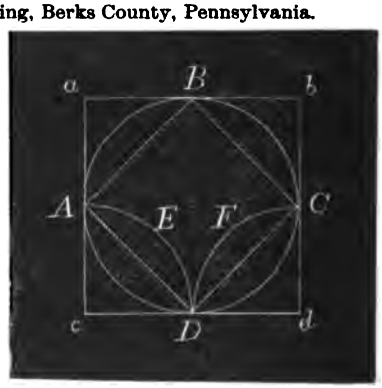
whence we easily find $$x=72\tfrac{1}{2} \text{ feet.}$$

Dr. Bachelder, William Hoover, K. S. Putnam and *W. L. Harvey* solved by the same method.

3.—Proposed by S. C. Gould, Manchester, Hillsborough County, New Hampshire.

It is required to enclose the exact area of a given square by means of circular arcs only.

I.—Solution by the Proposer; and F. P. Matz, B. E., B. S., Reading, Berks County, Pennsylvania.

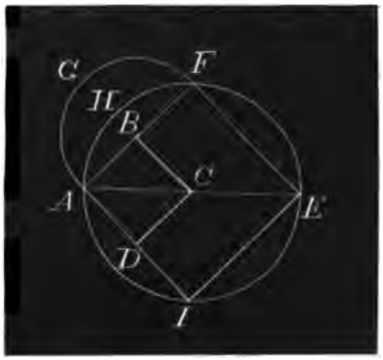


Let ABCD be the given square; draw the circumscribing circle and the circumscribing square, *abcd*, the corners of the given square being at the middle points of the sides of the circumscribed square. With centers *c* and *d* and radii equal to that of the circumscribing circle, draw the arcs AED, CFD, and the area of the pelicoid DEABCF is equal to the area of the square ABCD.

Proof.—Let S be the area of the square, s the area of one of the equal segments cut off from the circle by the square, and P the area of the pelicoid.

Then $$P=S-2s+2s=S.$$

II.—Solution by O. D. OATHOUT, Read, Clayton County, Iowa; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and WILLIAM HOOVER, Mathematical Editor of the *Wittenberger*, Bellefontaine, Logan County, Ohio.



Let ABCD be the given square, of which AC is a diagonal. With AC as a radius, describe the circle AFEI, and on AF as a diameter, describe the semicircle AGF. By the property of the lune of Hippocrates, the lune

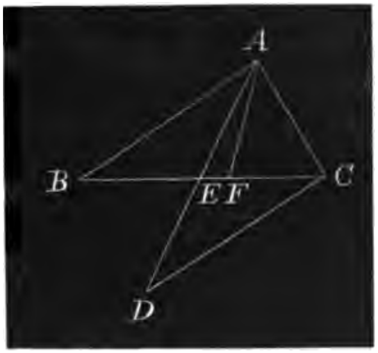
$$AGFH = \tfrac{1}{2}\triangle AFE = \tfrac{1}{4}\text{ square } AFEI = \text{square } ABCD.$$

This problem was also solved in a very ingenious manner by *B. F. Burleson* and *K. S. Putnam.*

4.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

The straight line drawn, bisecting any angle of a triangle, to the opposite side, is less than half the sum of the sides containing that angle.

I.—Solution by WILLIAM HOOVER, Mathematical Editor of the *Wittenberger*, Bellefontaine, Logan County, Ohio.



Let ABC be the triangle, AE the bisector of the base and AF the bisector of the vertical angle. Produce AE to D, making ED=AE, and draw CD. Then $AD < AC + CD$, or $2AE < AC + AB$,

$$\text{or } AE < \tfrac{1}{2}(AC+AB).$$

But AF is less than or equal to AE; hence

$$AF < \tfrac{1}{2}(AC+AB).$$

Good solutions given by *Dr. Bachelder* and *K. S. Putnam.*

II.—Solution by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

Let the line in question be x, then we have

$$x = \frac{c\sin B}{\cos\frac{1}{2}(B-C)} = \frac{ac\sin B}{(b+c)\sin\frac{1}{2}A} = \frac{2\sqrt{bcs(s-a)}}{b+c} = \sqrt{bc - \frac{bca^2}{(b+c)^2}}.$$

But this quantity is less than $\frac{1}{2}(b+c)$, since

$$bc - \frac{bca^2}{(b+c)^2} < \tfrac{1}{4}(b+c)^2$$

because $$bc[(b+c)^2 - a^2] < \tfrac{1}{4}(b+c)^4$$

or $$bc(a+b+c)(b+c-a) < \tfrac{1}{4}(b+c)^2(b+c)^2$$

as $$bc < \tfrac{1}{4}(b+c)^2 \text{ and } (a+b+c)(b+c-a) < (b+c)^2.$$

Mr. *Burleson's* solution was similar to this.

5.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

If 22 oxen and 28 cows eat 24 acres of grass in 18 weeks, and 20 oxen and 38 cows eat 30 aces of grass in 27 weeks, and 41 oxen and 26 cows eat 50 acres of grass in 60 weeks, how long will 40 acres of the same grass last 35 oxen and 14 cows, the grass in all cases growing uniformly?

Solution by THEO. L. DELAND, U. S. Treasury Department, Washington, D. C.; K. S. PUTNAM, Rome, Oneida County, New York; and ——.

Let x=the grass and its growth eaten by one ox from one acre in one week; y=the quantity eaten by one cow; g=the weekly growth on one acre, and t=the required time.

Then $18(22x+28y)$=what 22 oxen and 28 cows can eat in 18 weeks; and $24(1+18g)$=quantity of grass eaten from 24 acres in 18 weeks. By the problem these two expressions are equal; hence if we equate them, and extend the reasoning to the other conditions we will have:

$$18(22x+28y)=24(1+18g) \quad . \quad . \quad (1); \qquad 27(20x+38y)=30(1+27g) \quad . \quad . \quad (2);$$
$$60(41x+26y)=50(1+60g) \quad . \quad . \quad (3); \qquad t(35x+14y)=40(1+tg) \quad . \quad . \quad (4).$$

From (1), (2) and (3) we get

$$33x+42y-36g=2 \quad . \quad . \quad . \quad . \quad . \quad (5), \qquad 90x+171y-135g=5 \quad . \quad . \quad . \quad . \quad (6),$$
$$246x+156y-300g=5 \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Solving these equations, we find

$$x=\frac{5}{54}, \qquad y=\frac{5}{108}, \qquad g=\frac{1}{12}.$$

From (4),

$$t=\frac{40}{35x+14y-40g}=72 \text{ weeks.}$$

The solution by *W. L. Harvey* was similar. An ingenious solution was furnished by *B. F. Burleson.*

6.—Proposed by THEO. L. DELAND, U. S. Treasury Department, Washington, D. C.

Four men, A, B, C and D, agreed to reap a circular field of grain in 28 days for $249.36, and divide the money among them in proportion to the actual service each rendered. The first, second, third and fourth quadrants contained, respectively, wheat, oats, barley and rye. To the end of the 7th day A cut wheat, B oats, C barley and D rye. They each then advanced one quadrant, and to the end of the 13th day A cut oats, B barley, C rye and D wheat; they again advanced one quadrant, and cut to the end of the 18th day; again they advanced one quadrant each and so worked to the end of the 22d day, and finished the wheat; again they advanced one quadrant each and finished the oats at the end of three days more; they again advanced one quadrant each and finished the barley at the end of the 27th day; and then advanced one quadrant each and completed the work according to contract. Required—the time for each to reap the field, and an equitable division of the money, provided one quadrant is as difficult as another.

Solution by Mrs. ANNA T. SNYDER, Orange, Fayette County, Indiana; JAMES MCLAUGHLIN, Mantorville, Dodge County, Minnesota; and W. L. HARVEY, Maxfield, Penobscot County, Maine.

Let w=part of the whole work A does in a day,
x= " " " B " "
y= " " " C " "
z= " " " D " "

By the conditions of the problem,

$$7w+4x+5y+6z=\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$6w+10x+4y+5z=\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$5w+8x+10y+4z=\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad (3),$$
$$4w+6x+8y+10z=\tfrac{1}{4} \quad . \quad . \quad . \quad . \quad . \quad (4).$$

From these equations we easily find

$$w=\frac{174}{8312}, \quad x=\frac{49}{8312}, \quad y=\frac{56}{8312}, \quad z=\frac{64}{8312}.$$

$\frac{1}{w}=\frac{8312}{174}=47\frac{67}{87}$, the number of days in which A can cut all the grain;

$\frac{1}{x}=\frac{8312}{49}=169\frac{31}{49}$, " " " " B " " "

$\frac{1}{y}=\frac{8312}{56}=148\frac{3}{7}$, " " " " C " " "

$\frac{1}{z}=\frac{8312}{64}=129\frac{7}{8}$, " " " " D " " ".

A worked 22 days and his share of the money is $\$249.36\times\frac{174}{8312}\times22=\$114.84$;

B " 28 " " " " " $\$249.36\times\frac{49}{8312}\times28=\$41.16$;

C " 27 " " " " " $\$249.36\times\frac{56}{8312}\times27=\$45.36$;

D " 25 " " " " " $\$249.36\times\frac{64}{8312}\times25=\$48.00$.

Solved also in an elegant manner by the *Proposer* and *B. F. Burleson.*

7.—Proposed by Prof. DAVID TROWBRIDGE, M. A., Waterburg, Tompkins County, New York.

If A, B, C be the angles of a triangle, prove that

$$\frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}+\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}+\frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=1.$$

I.—Solution by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

We have $\cot A+\cot B=\frac{c^2}{2K}$, $\cot\frac{1}{2}A+\cot\frac{1}{2}B=\frac{sc}{K}$;

hence
$$\frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}=\frac{c}{2s}.$$

Similarly,
$$\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}=\frac{b}{2s} \text{ and } \frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=\frac{a}{2s}.$$

$$\therefore \frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}+\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}+\frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=\frac{c+b+a}{2s},=1, \text{ since } 2s=a+b+c.$$

II.—Solution by ——; and the PROPOSER.

Let p_1, p_2, p_3 be the perpendiculars let fall from A, B and C respectively upon a, b and c; and r be the radius of the inscribed circle.

Then
$$p_1(\cot B+\cot C)=a=r(\cot\frac{1}{2}B+\cot\frac{1}{2}C);$$
$$\therefore \frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=\frac{r}{p_1}.$$

Similarly,
$$\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}=\frac{r}{p_2}, \quad \frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}=\frac{r}{p_3};$$

$$\therefore \frac{\cot A+\cot B}{\cot\frac{1}{2}A+\cot\frac{1}{2}B}+\frac{\cot A+\cot C}{\cot\frac{1}{2}A+\cot\frac{1}{2}C}+\frac{\cot B+\cot C}{\cot\frac{1}{2}B+\cot\frac{1}{2}C}=r\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)=r\left(\frac{1}{r}\right)=1.$$

Solved also by *B. F. Burleson, James McLaughlin* and *David Wickersham.*

8.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In an equilateral triangle ABC, lines drawn from each vertex to the opposite side, dividing that side so that the ratio of the side to one of the segments is n, form by their intersections another equilateral triangle; determine the ratio of the areas of these triangles.

Solution by Dr. S. F. BACHELDER, South Boston, Massachusetts; and B. F. BURLESON, Oneida Castle, Oneida County, New York.

Draw AE, BG and CE, making

$$AD = BE = CG = \frac{AB}{n}.$$

Put x = the required ratio, then

$$x = \frac{\triangle FIH}{\triangle ABC} = \frac{(FH)^2}{(AC)_2} \quad . \quad . \quad . \quad . \quad (1).$$

Put $a = AB$, then $\frac{a}{n} = AD$.

By trigonometry, $\cos CAD = \frac{(AC)^2 + (AD)^2 - (CD)^2}{2AC.AD}$ . . . . (2).

But $\cos CAD = \frac{1}{2}$. Substituting in (2), $\dfrac{a^2 + \frac{a^2}{n^2} - (CD)^2}{\frac{2a^2}{n}} = \frac{1}{2}$;

whence $CD = \frac{a\sqrt{n^2 - n + 1}}{n}$, which put $= b$.

The triangle ADF is similar to CDA, consequently

CD : CA :: AD : AF or CH . . . . (3),

and CD : AD :: AD : FD . . . . (4).

Substituting known values in (3) and (4),

$$CH = \frac{a^2}{bn},\ FD = \frac{a^2}{bn^2},\ HF = CD - (CH + FD) = \frac{b^2n^2 - a^2(n+1)}{bn^2}.$$

Substituting in (1),

$$x = \frac{\triangle FIH}{\triangle ABC} = \left(\frac{b^2n^2 - a^2(n+1)}{abn^2}\right)^2 = \frac{(n-2)^2}{n^2 - n + 1}.$$

Good solutions given by the *Proposer*, *E. B. Seitz* and *K. S. Putnam*.

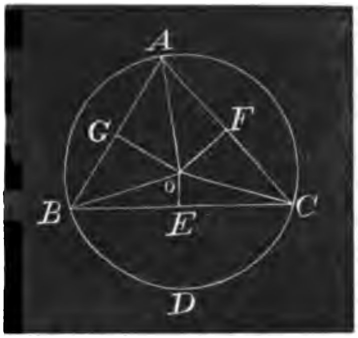
9.—Proposed by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

Show that in any plane triangle, $\dfrac{a \sin A + b \sin B + c \sin C}{a \cos A + b \cos B + c \cos C} = R\left(\dfrac{a^2 + b^2 + c^2}{abc}\right)$.

Solution by DAVID WICKERSHAM, Wilmington, Clinton County, Ohio; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and WILLIAM HOOVER, Mathematical Editor *Wittenberger*, Bellefontaine, Logan County, Ohio,

Let ABC be the triangle; and let ABDC be a circle circumscribing the triangle, and O its center; and join OA, OB and OC, and draw OE, OF and OG respectively perpendicular to BC, AC and AB.

Let R = AO or BO; then BE $= R \sin A$, and BC $= a = 2R \sin A$, and $\sin A = \frac{a}{2R}$,



and OE$=R$ cosA, and cosA$=\frac{OE}{R}$. a sinA$=\frac{a^2}{2R}$, and in the same way b sinB$=\frac{b^2}{2R}$ and c sinC$=\frac{c^2}{2R}$; and

$$a \sin A + b \sin B + c \sin C = \frac{a^2+b^2+c^2}{2R} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

And a cos A$=a\times\frac{OE}{R}$; but $a\times\frac{OE}{R}$ is double the area of $\triangle$COB$\div R$; in the same way b cosB$=$double the area of $\triangle$AOC$\div R$; and c cosC$=$double the area of $\triangle$AOB$\div R$. But double the area of $\triangle$ABC is$=ab$ sinC$=\frac{abc}{2R}$; double the area of $\triangle$ABC$\div R$ is$=\frac{abc}{2R^2}$; and a cosA$+b$ cosB$+c$ cosC$=\frac{abc}{2R^2}$ . . . . (2); and dividing (1) by (2), $\frac{a \sin A + b \sin B + c \sin C}{a \cos A + b \cos B + c \cos C} = R\left(\frac{a^2+b^2+c^2}{abc}\right)$.

Solved also in an elegant manner by the *Proposer* and *B. F. Burleson.*

10.—Proposed by Prof. H. A. Wood, M. A., Principal Coxsackie Academy, Coxsackie, Greene Co., N. Y.

A hollow paraboloid, depth 4 feet, stands on its vertex with axis vertical. After a shower the depth of water in the paraboloid, measured on the axis, was 2 inches. What was the uniform depth of the rainfall, the radius of the top of the paraboloid being 3 feet?

I.—Solution by B. F. Burleson, Oneida Castle, Oneida County, New York.

Put D=entire depth of paraboloid,$=$48 inches,
d=depth of rain caught,$=$2 inches,
and R=radius of the top of the paraboloid,$=$36 inches.

Let r=radius of the top of the water in the paraboloid, and V=volume of the same. Then, as the squares of ordinates to the axis in all parabolas are to each other as their corresponding abscissas, (*Loomis' Conic Sections*, Prop. VIII, Cor. 1,) $R^2 : r^2 :: D : d$; $\therefore$ $r^2=\frac{R^2d}{D}$. The volume of the paraboloid of rain caught being one-half its circumscribing cylinder (*Loomis' Calculus*, Art. 328,) we have $V=\frac{\pi R^2d^2}{2D}$. Obviously, therefore, $\frac{\pi R^2d^2}{2D}\div\pi R^2=\frac{d^2}{2D}$, equal, by restoring numerical values, $\frac{1}{24}$ of an inch,$=$the uniform depth of the rainfall required.

As the general expression giving the uniform depth of the rainfall does not contain the factor R we infer that the condition giving the radius of the top of the paraboloid is superfluous—the answer being easily obtainable without its use, being constant for every value that may be given to R.

II.—Solution by Marcus Baker, U. S. Coast Survey Office, Washington, D. C.; D. J. McAdam, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania; and Orlando D. Oathout, Read, Clayton County, Iowa.

The general equation of the parabola, $y^2=2px$, gives for the parabola which generates the paraboloid of our problem $2p=\frac{y^2}{x}=\frac{9}{4}$; whence $y^2=\frac{9}{4}x$.

For the volume of a paraboloid of revolution, we have $V = \frac{1}{2}\pi y^2 x$, and hence when $x = 2$ inches, $= \frac{1}{6}$ foot, $y^2 = \frac{3}{8}$ and therefore $V = \frac{\pi}{32}$, which is the amount of rainfall upon the surface 9π; hence the uniform depth of the rainfall is, in inches,

$$\frac{12\pi}{32} \div 9\pi = \frac{1}{24}.$$

This problem was also solved by Messrs. *Bachelder*, *Hoover*, *Matz* and *Putnam*.

11.—Proposed by B. F. Burleson, Oneida Castle, Oneida County, New York.

The sides of a quadrilateral field are, respectively, and in order, 17, 35, 40 and 45 rods in length. Determine the length of the straight line which in passing through a point equally distant from all its angles, shall divide the field into two equal parts.

Solution by the Proposer; and David Wickersham, Wilmington, Clinton County, Ohio.

Let ABCD represent the quadrilateral, and the point O the center of its circumscribed circle. Let EOF represent the straight division line required, the two ends E and F of which must fall, it may be easily determined, on the lines AD and BC of the quadrilateral, respectively. Draw the diagonals AC, BD; also join AO and BO, producing the latter until it meets the side AD in M.

Put AB $= 40 = a$, BC $= 35 = b$, CD $= 17 = c$ and DA $= 45 = d$. Let $x =$ BD, $y =$ AC, $r =$ AO and $A =$ area of the quadrilateral ABCD.

By Geometry, $xy = ac + bd$ . . (1). Also $bc + ad : cd + ab :: x : y$ . . . (2).

Hence $$x = \sqrt{\left[\frac{(ac+bd)(ab+cd)}{ad+bc}\right]} = 45.149138 \text{ rods},$$

and $$y = \sqrt{\left[\frac{(ac+bd)(bc+ad)}{ab+cd}\right]} = 49.945582 \text{ rods}$$

Also, $$r = \tfrac{1}{4}\sqrt{\left[\frac{(ab+cd)(ac+bd)(bc+ad)}{(s-a)(s-b)(s-c)(s-d)}\right]} = 25.148613 \text{ rods},$$

and $\frac{1}{2}A = \frac{1}{2}\sqrt{[(s-a)(s-b)(s-c)(s-d)]} = 537.467578$ square rods, where $2s = a+b+c+d$.

By Trigonometry we now easily determine the following angles: ABC $= 83°\ 13'\ 21''$, BAD $= 63°\ 51'\ 3''$ and ABM $= 37°\ 19'\ 7''$. We also determine further, by Trigonometry, that BM $= 36.59924$ rods, OM $= 11.45062$ rods, AM $= 24.71801$ rods, the angle EMO $= 78°\ 49'\ 50''$ and the angle OBF $= 45°\ 54'\ 14''$. We now find the area of the triangle ABM $= 443.762137$ square rods, which lacks just 93.705441 square rods of fulfilling the conditions of the problem. Supposing EF to represent the true division line, it will be evident that the difference in the areas of the two triangles BOF and EOM must be 93.705441 square rods

Put $m =$ OM $= 11.45062$, $\frac{1}{2}g = 93.705441$ square rods, $\alpha = \angle$ OBF $= 45°\ 54'\ 14''$ and $\beta = \angle$ EMO $= 78°\ 49'\ 50''$. Let $z =$ OF, $w =$ OE and $\varphi = \angle$ EOM or BOF. In the triangle BOF, $\sin\alpha : z :: \sin(\alpha+\varphi) : r$; $\therefore$ $z = \frac{r \sin\alpha}{\sin(\alpha+\varphi)}$. In the triangle EOM, $\sin\beta : w :: \sin(\beta+\varphi) : m$; $\therefore$ $w = \frac{m \sin\beta}{\sin(\beta+\varphi)}$. Now area BOF $= \frac{1}{2}rz \sin\varphi$, and area EOM $= \frac{1}{2}mw \sin\varphi$, hence $rz \sin\varphi - mw \sin\varphi = g$ . . . . (3).

By substituting in (3) the values found for z and w, and reducing, we have

$$\frac{r^2}{\cot\varphi+\cot\alpha}-\frac{m^2}{\cot\varphi+\cot\beta}=g \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4).$$

In equation (4) we can easily find the value of $\cot\varphi$, and hence the angle φ or EOM, which is 36° 20′ 17″. Having found the angle EOM or BOF we can find the remaining parts of the two triangles EOM and BOF. The parts EO in the one and OF in the other, when added together, give EF$=$30.63992 rods.

Solved also by *Dr. Bachelder* and *James McLaughlin*.

12.—Proposed by James McLaughlin, Mantorville, Dodge County, Minnesota.

A gentleman has a rectangular garden in latitude 40° 41′ north, the diagonal of which is a meridian line. At the south corner of the garden stands a perpendicular pine tree, the shade of which at noon, when the days and nights are equal, extends to a small rivulet which comes in at the east corner of the garden and runs due west through it, crossing the said meridian line 240 feet from the north corner of the garden, and going out of it 135 feet from the pine tree. Required the area of the garden and the hight of the tree.

I.—Solution by E. B. Seitz, Greenville, Darke County, Ohio; B. F. Burleson, Oneida Castle, Oneida Co., New York; and the Proposer.

Let NESW be the garden, NS the meridian line, E the east corner, and EHK the rivulet.

Put NH$=a=$240 feet, SK$=b=$135 feet, SH$=x$, $h=$hight of the tree, and $\lambda=$41° 40′, the latitude. Then from the right angled triangles we have

SH : NH :: SK : NE and SN : SK :: NE : SH.

Taking the products of the extremes and the means, we have $(\text{SH})^2.\text{SN}=(\text{SK})^2.\text{NH}$, or $x^3+ax^2=ab^2$.

Restoring the numbers, we have $x^3+240x^2=4374000$, whence $x=111.544738$ feet.

∴ area NESW$=$NS . EH$=(a+x)\sqrt{(ax)}=57518.926$ square feet$=$1.320453 acres, and $h=x\cot\lambda=125.342$ feet.

Solved also by *Dr. Bachelder* and *David Wickersham*.

II.—Solution by Dr. Samuel Hart Wright, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, Yates County, New York.

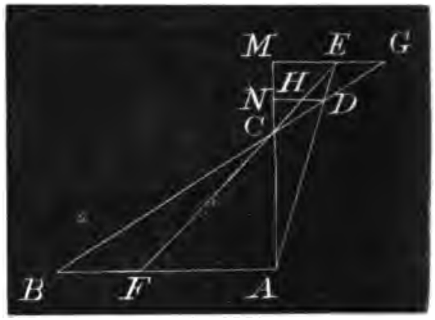
Let $x=$a side and y an end of the garden, $a=135$, $b=240$. Then shadow$=(x^2+y^2)^{1/2}-b$, and $y:x::a:y$; ∴ $y^2=ax$. $b:x::y:(x^2+y^2)^{1/2}$; ∴ $y^2=\frac{a^2b^2}{x^2-b^2}=ax$, and $x^3-b^2x=ab^2$, whence $x=290.467$ and $y=198.013$. Area$=$57519.1 square feet$=$1.32046 acres. Shadow$=111.545=b_1$. Declination of sun's upper limb$=+16'=r$.

Hight of tree$=b_1\cot(\lambda-r)=b_1\cot(41°\ 10'-16')=128.765$ feet.

13.—Proposed by E. P. Norton, Allen, Hillsdale County, Michigan.

In a triangle ABD the base AB is 40 rods. A line AC drawn from the angle A, and perpendicular to AB, intersecting BD in C, is 9 rods. If a point F be taken in AB, 10 rods from B, and a line drawn from this point through C, and produced, it will cut AD produced in a point E, 5 rods from D. Required the sides of the triangle

Solution by E. B. Seitz, Greenville, Darke County, Ohio.



Draw DHN perpendicular to AC produced, and draw GEM parallel to DN, meeting BD produced in G.

Let $AB=a=40$ rods, $BF=b=10$ rods, $AC=c=9$ rods, $DE=d=5$ rods, $EG=x$, and $HD=y$. Then from similar right angled triangles we find $CM=\frac{cx}{b}$, $CN=\frac{cy}{b}$, $EM=\frac{(a-b)x}{b}$, $DN=\frac{ay}{b}$ and $AM:AN::EM:DN$,

$$\text{or } b+x : b+y :: (a-b)x : ay \quad \ldots \quad (1).$$

We also have $(CM-CN)^2+(EM-DN)^2=(DE)^2$, or $c^2(x-y)^2+[(a-b)x-ay]^2=b^2d^2$. (2).

From (1) $y=\frac{(a-b)x}{a+x}$. Substituting this value of y in (2), and reducing, we have

$$[(a-b)^2+c^2]x^4+2bc^2x^3+(c^2-d^2)b^2x^2-2ab^2d^2x-a^2b^2d^2=0 \quad \ldots \quad (3).$$

Restoring the numbers in (3), we have

$$981x^4+1620x^3+5600x^2-200000x-4000000=0 \quad \ldots \quad (4).$$

Solving (4), we find $x=8.172543366$ rods. $\therefore$ $y=5.089544455$ rods,

$BD=(a^2+c^2)^{\frac{1}{2}}+\frac{y}{b}(a^2+c^2)^{\frac{1}{2}}=61.86713226$ rods, and $AD=\frac{d(b+y)}{x-y}=24.47218584$ rods.

Good solutions were given by Messrs. *Bachelder*, *Burleson*, *McLaughlin* and the *Proposer*.

14.—Proposed by Prof. Horatio M. Bloomfield, Reading, Berks County, Pennsylvania.

Given $w^2+y^2=c$, $z^2+y^2=b$, $z^2+x^2=a$, $x+y=w+z$; to find the value of x by quadratics.

Solution by B. F. Burleson, Oneida Castle, Oneida County, New York; and F. P. Matz, B. E., B. S., Reading, Berks County, Pennsylvania.

Let $s=x+y=w+z$ and we evidently have the three equations

$z^2+(s-y)^2=a$ . . (5), $\quad z^2+y^2=b$ . . (6), $\quad y^2+(s-z)^2=c$ . . (7).

Eliminating y and z from (5), (6) and (7), we obtain

$$2s^4-(2a+2c)s^2=2bc+2ab-(a^2+2b^2+c^2) \quad \ldots \quad (8).$$

Resolving (8) s, or $x+y,=\pm\{\frac{1}{2}(a+c)\pm\frac{1}{2}[4ab+4bc+2ac-a^2-c^2-4b^2]^{\frac{1}{2}}\}^{\frac{1}{2}}$ . . (9).

Subtracting (2) from (3), $\quad x^2-y^2=a-b \quad \ldots \quad (10).$

Solving (9) and (10), we find $x=\dfrac{3a-2b+c\pm[4ab+4bc+2ac-a^2-2b^2-c^2]^{\frac{1}{2}}}{\pm2\{2(a+c)\pm2[4ab+4bc+2ac-a^2-2b^2-c^2]^{\frac{1}{2}}\}^{\frac{1}{2}}}$.

Unfinished solutions given by *Marcus Baker* and *William Hoover*.

15.—Proposed by Henry Heaton, B. S., Superintendent of Schools, Sabula, Jackson County, Iowa.

If n persons meet by chance, what is the probability that they all have the same birthday, supposing every fourth year a leap year?

Solution by the PROPOSER; and K. S. PUTNAM, Rome, Oneida County, New York.

Since the twenty-ninth of February occurs but once in every 1461 days, the probability that it is the birthday of any given person is $\frac{1}{1461}$, and the probability that it is the birthday of n persons is $\frac{1}{(1461)^n}$. Since any other given date occurs 4 times in 1461 days, the probability that *it* is the birthday of n persons is $\left(\frac{4}{1461}\right)^n$, and since there are 365 such dates, the required probability is

$$\frac{1}{(1461)^n}+365\left(\frac{4}{1461}\right)^n=\frac{1+365(4)^n}{(1461)^n}.$$

SOLUTIONS OF SENIOR PROBLEMS,

Proposed in No. 1.

16.—Proposed by Prof. DAVID TROWBRIDGE, M. A., Waterburg, Tompkins County, New York.

Required the expansion of θ^2 in a series of cosines of multiples of θ, thus:

$$\theta^2=A_0+A_1\cos\theta+A_2\cos2\theta+\ \ .\ \ .\ \ .\ \ +A_n\cos n\theta+\ \ .\ \ .\ \ .$$

Solution by the PROPOSER; and Prof. H. T. J. LUDWICK, Mount Pleasant, Cabarrus Co., North Carolina.

We have

$$\int_{-\pi}^{+\pi}\theta^2d\theta=\tfrac{2}{3}\pi^3=\int_{-\pi}^{+\pi}[A_0d\theta+A_1\cos\theta d\theta+A_2\cos2\theta d\theta+\ \ .\ \ .\ \ ]=2\pi A_0,\quad\therefore\quad A_0=\tfrac{1}{3}\pi^2.$$

Also, $$\int_{-\pi}^{+\pi}\theta^2\cos m\theta d\theta=\left[\frac{\theta^2}{m}\sin m\theta+\frac{2\theta}{m^2}\cos m\theta-\frac{2}{m^3}\sin m\theta\right]_{-\pi}^{+\pi}=+\frac{4\pi}{m^2}(-1)^m,$$

$$=\int_{-\pi}^{+\pi}d\theta\cos m\theta[A_0+A_1\cos\theta+\ \ .\ \ .\ \ +A_m\cos m\theta+\ \ .\ \ .\ \ ]=A_m\pi;$$

$\therefore\quad A_m=+\frac{4(-1)_m}{m^2}$. We hence have $\theta^2=\frac{1}{3}\pi^2-4\left[\cos\theta-\frac{\cos2\theta}{2^2}+\frac{\cos3\theta}{3^2}-\ \ .\ \ .\ \ \right]$.

Cor.—If $\theta=0$, $\frac{\pi^2}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ .\ .\ ;$

if $\theta=\pi$, $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ \ .\ \ .\ \ .$

Solved also by *G. M. Day.*

17.—Proposed by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

The polar with respect to a triangle of its orthogonal center is the radical axis of the circumscribing circle, the nine-point circle and the circle with respect to which the triangle is self-conjugate.

Solution by the PROPOSER.

The equation to the nine point circle (*Salmon*, p. 127,) may be thrown into the form

$$(\alpha\cos A+\beta\cos B+\gamma\cos C)(\alpha\sin A+\beta\sin B+\gamma\sin C)-2(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)=0;$$

that of the circle with respect to which the triange is self-conjugate (*Salmon*, p. 243,) may be written

$$(\alpha\sin A+\beta\sin B+\gamma\sin C)(\alpha\cos A+\beta\cos B+\gamma\cos C)-(\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C)=0.$$

Hence it appears that their common radical axis with the circle

$$\beta\gamma\sin A+\gamma\alpha\sin B+\alpha\beta\sin C=0 \quad\text{is}\quad \alpha\cos A+\beta\cos B+\gamma\cos C=0.$$

Solved also in an elegant manner by *G. M. Day*.

18.—Proposed by I. H. TURRELL, Cumminsville, Hamilton County, Ohio.

Suppose n fixed lines, radiating from a fixed point, are touched two and two by n circles of given radii; that is, that each line touches two of the circles, and each circle touches two of the lines. The order of arrangement being known, it is required to construct geometrically a polygon of n sides, whose vertices shall lie on the n lines, and whose sides shall touch the given circles.

Solution by the PROPOSER.

Let X, X_1, X_2, . . X_{n-1} be the n lines given in position, and Y_1, Y_2. . . . Y_n the given circles. Place Y_1 touching the lines X, X_1; Y_2 touching the lines X_1, X_2; . . . . Y_n touching the lines X_{n-1}, X.

From any point a in X draw a tangent to Y_1, meeting X_1 in some point a_1; from a_1 draw a tangent to Y_2, meeting X_2 in a_2, and so on, until a point a_{n-1} in X_{n-1} is determined. From a_{n-1} draw a tangent to Y_n meeting X in a_n. If a, a_n are coincident the polygon is constructed. In like manner take two other points b, c in X, and obtain the corresponding points b_n, c_n.

Suppose p to be the required point in X; then p, p_n are coincident. Now the four tangents to Y_1, viz: aa_1, bb_1, cc_1, pp_1, are cut by the tangents X, X_1; hence (*Mulcahy's Modern Geometry*, p. 39,) the anharmonic ratio $a, b, c, p=a_1, b_1, c_1, p_1$. Similarly $a_1, b_1, c_1, p_1=a_2, b_2, c_2, p_2$, and so on; hence $abcp=a_nb_nc_np_n$. Therefore in X we have six known points, a, b, c, a_n, b_n, c_n, to find a seventh point p in the same line such that $abcp=a_nb_nc_np_n$.

This is an elementary problem in the Modern Geometry (See *Mulcahy*, p. 23,) and gives two positions for y. Hence the original problem admits of two solutions, when the lines aa_1, bb_1. &c., are *direct* tangents to the given circles. If some of the tangents are *transverse*, the number of solutions is vastly increased, and it is evident that numerous impossible cases may arise.

19.—Proposed by Dr. S. H. WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, Yates County, New York.

In latitude $\lambda=42°$N., when the sun's declination north$=\delta=20°$, the sun shone on the top of a high peak in a mountain $\beta=4$ minutes before it shone on the plane of the horizon. How high was the peak?

I.—Solution by E. B. SEITZ, Greenville, Darke County, Ohio; and GEORGE EASTWOOD, Saxonville, Middlesex County, Massachusetts.

Let Z be the zenith of the mountain peak, S the place of the sun's center when it first shone on the top of the peak, S′ the place when it first shone on the plane of the horizon, N the north pole, and let ZH be perpendicular to SN produced. Put $NZ=\frac{1}{2}\pi-\lambda$, $SN=S'N=\frac{1}{2}\pi-\delta$, $S'Z=\frac{1}{2}\pi+f+r=\psi$, $f=34'$, the refraction of

light at the horizon, $r=16'$, the sun's radius, $ZH=a$, $HN=b$, $SH=c$, $SZ=d$, $\angle S'NZ=\theta$, $\angle SNZ=\varphi$, $\angle SNS'=\frac{1}{2}\beta=1°$, $R=3956$ miles, the radius of the earth; $h=$ the hight of the peak. Then in the spherical triangle S'NZ we find

$$\theta=2\tan^{-1}\left(\frac{\sin\frac{1}{2}(\lambda-\delta+\psi)\sin\frac{1}{2}(\delta-\lambda+\psi)}{\cos\frac{1}{2}(\psi-\lambda-\delta)\cos\frac{1}{2}(\lambda+\delta+\psi)}\right)=110°\ 23'\ 54''.$$

$\therefore\ \varphi=\theta+\frac{1}{2}\beta=111°\ 23'\ 54''$. In the triangle ZHN we have $a=\sin^{-1}(\cos\lambda\sin\varphi)=43°\ 46'\ 55''$, and $b=\cos^{-1}(\sin\lambda\sec a)=22°\ 3'\ 29''$; $\therefore\ c=\frac{1}{2}\pi-\delta+b=92°\ 3'\ 29''$.

In the triangle SZH we have $d=\cos^{-1}(\cos a\cos c)=91°\ 29'\ 8\ 7''$.

$\therefore\ h=R\sec(d-\psi)-R=2R\sin^2\frac{1}{2}(d-\psi)\sec(d-\psi)=1354.215$ feet.

Solved in a similar manner by *D. J. McAdam.*

II.—Solution by the Proposer.

Let the zenith distance of the sun's center when its upper limb rises on the plane $=(90°+$refraction $34'+$sun's radius $16')=90°\ 50'=z$; hour angle then $=P$; zenith distance when rising on the peak $=z'$, and hour angle then $=P'$; $R=3956\times5280$ feet. Then

$P=2\sin^{-1}\sqrt{[\sin\frac{1}{2}(\lambda+z-\delta)\sin\frac{1}{2}(\delta-\lambda+z)\sec\lambda\sec\delta]}=110°\ 23'\ 54'$.

$P'=P+\frac{1}{2}\beta=111°\ 23'\ 54''$, and $z'=\cos^{-1}[\sin(\delta+\varphi)\sin\lambda\sec\varphi]=91°\ 29'\ 8.7''$, where $\tan\varphi=\cot\lambda\cos P'$. Dip of horizon $=z'-z=39'\ 8.7''=A$. Then hight of peak $=R(\sec A-1)=2R\sin^2\frac{1}{2}A\sec A=1354.215$ feet.

20.—Proposed by William Hoover, Mathematical Editor *Wittenberger*, Bellefontaine, Logan Co., Ohio.

The weights W_1 and W_2 are suspended, θ degrees apart, from the circumference of a vertical circle free to move about its center. Find the angle made by the radii drawn from the points of suspension with the horizontal through the center of the circle when the system is in equilibrium.

Solution by Dunlap J. McAdam, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania; and E. B. Seitz, Greenville, Darke County, Ohio.

Let $\chi=$angle which the radius drawn from W_1 makes with the horizontal when in equilibrium, then $\pi-\chi-\theta=\varphi$, the other required angle, and taking moments about the center of the circle we have $W_1\cos\chi=W_2\cos(\pi-\chi-\theta)$, which gives $\chi=\tan^{-1}\left(\frac{W_1+W_2\cos\theta}{W_2\sin\theta}\right)$. Similarly we find $\varphi=\tan^{-1}\left(\frac{W_2+W_1\cos\theta}{W_1\sin\theta}\right)$.

Solved in a similar manner by *Messrs. Hoover* and *Matz*, and geometrically by *G. M. Day.*

21.—Proposed by Dr. David S. Hart, M. A., Stonington, New London County, Connecticut.

Required five numbers whose sum shall be a biquadrate, and the sum of any four of them a square.

Solution by the Proposer.

Let v, w, x, y, z represent the numbers. Then $v+w+x+y+z=$biquadrate$=m^4$ (1),

$v+w+x+y=\square=n^2$, $v+w+x+z=\square=p^2$, $v+w+y+z=\square=q^2$,

$v+x+y+z=\square=r^2$, $w+x+y+z=\square=s^2$.

Adding the last four equations, and dividing by 4, we have

$$v+w+x+y+z=\tfrac{1}{4}(n^2+p^2+q^2+r^2+s^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

From this, subtracting the last four equations, we have $v=\frac{1}{4}(n^2+p^2+q^2+r^2-3s^2)$, $w=\frac{1}{4}(n^2+p^2+q^2+s^2-3r^2)$, $x=\frac{1}{4}(n^2+p^2+r^2+s^2-3q^2)$, $y=\frac{1}{4}(n^2+q^2+r^2+s^2-3p^2)$, $z=\frac{1}{4}(p^2+q^2+r^2+s^2-3n^2)$. Equating the right-hand members of (1) and (2) and then multiplying by 4, $\quad n^2+p^2+q^2+r^2+s^2=4m^4$.

Let $n=am$, $p=bm$, $q=cm$, $r=dm$, $s=em$; then, substituting and dividing by m^2, we have $\quad a^2+b^2+c^2+d^2+e^2=4m^2$.

Let $a=m-A$, $b=m-B$, $c=m-C$, $d=m-D$; then, substituting, canceling, transposing, &c., we have $m=\dfrac{A^2+B^2+C^2+D^2+e^2}{2(A+B+C+D)}$, in which expression, in order that the values of v, w, x, y, z may be positive, A, B, C, D must be taken consecutive numbers in the natural series, or nearly so, and e must be$=2(A+B+C+D)$, or nearly so.

Let $A=1$, $B=2$, $C=3$, $D=4$; then $e=20$, $m=\dfrac{43}{2}$, $a=\dfrac{41}{2}$, $b=\dfrac{39}{2}$, $c=\dfrac{37}{2}$, $d=\dfrac{35}{2}$, or multiplying all these quantities by 2, $m=43$, $a=41$, $b=39$, $c=37$, $d=35$, $e=40$; whence $n=41\times43$, $p=39\times43$, $q=37\times43$, $r=35\times43$ and $s=40\times43$.

Then by substitution we have $v=249\times(43)^2=460401$, $w=624\times(43)^2=1153776$, $x=480\times(43)^2=887520$, $y=328\times(43)^2=606472$, $z=168\times(43)^2=310632$

From this solution we see how n numbers can be found whose sum is a biquadrate, and the sum of any $n-1$ of them a square.

22.—Proposed by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

An ellipsoid, axes $2a$, $2b$, $2c$, is intersected by a cylinder, radius b, the axis of the cylinder coinciding with the axis $2c$ of the ellipsoid. Find the volume common to both.—From the *Normal Monthly*, vol. III, p. 96.

Solution by E. B. SEITZ, Greenville, Darke County, Ohio; and the PROPOSER.

The equation to the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ . . . (1), and the equation to the cylinder is $x^2+y^2=b^2$ . . . (2). The volume common to the ellipsoid and cylinder is

$$V=8\int\int\int dxdydz=8\int\int zdxdy=8c\int\int\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dxdy.$$

The limits of x are 0 and $\sqrt{(b^2-y^2)}$; of y, 0 and b;

$$\therefore\quad V=8c\int\int\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}}dxdy=4ac\int_0^b\left[\left(1-\frac{y^2}{b^2}\right)\sin^{-1}\left(\frac{b}{a}\right)+\right.$$

$$\left.\frac{b}{a^2}(a^2-b^2)^{\frac{1}{2}}\left(1-\frac{y^2}{b^2}\right)\right]dy=\frac{8}{3}abc\left[\sin^{-1}\left(\frac{b}{a}\right)+\frac{b(a^2-b^2)^{\frac{1}{2}}}{a^2}\right].$$

23.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find the least integral values of x and y that will satisfy the equation

$$x^2-9781y^2=1.$$

Solution by the Proposer.

Put $A=9781$, then $\sqrt{A}=\sqrt{(9781)}=r+\cfrac{1}{u_1+\cfrac{1}{u_2+\cfrac{1}{u_3+\text{etc.},}}}$

where r is the integral part of $\sqrt{A}$. The last quotient of every complete period is $2r$. Let m be the number of quotients in a complete period, and $\frac{p_m}{q_m}$ the last convergent in the first period; then, when m is even, $x=p_m$, $y=q_m$, and when m is odd, $x=p_{2m}$, $y=q_{2m}$. Let u_n, u_{n+1} be any two consecutive partial quotients; then

$$\frac{r+a_n}{b_n}=u_n+\text{etc.},\quad \frac{r+a_{n+1}}{b_{n+1}}=u_{n+1}+\text{etc.};$$

$$a_0=0,\ b_0=1;\ a_1=r,\ b_1=A-r^2;\ u_0=r,\ u_1=\frac{2r}{A-r^2};\ a_{n+1}=u_nb_n-a_n,\ b_{n+1}=\frac{A-a^2_{n+1}}{b_n}.$$

If $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$ be any two consecutive convergents and u_{n+1} the quotient corresponding to $\frac{p_{n+1}}{q_{n+1}}$, then $\frac{p_1}{q_1}=\frac{r}{1}$, $\frac{p_2}{q_2}=\frac{ru_1+1}{u_1}$, ... $\frac{p_{n+2}}{q_{n+2}}=\frac{u_{n+1}p_{n+1}+p_n}{u_{n+1}q_{n+1}+q_n}$.

The partial quotients are easily found to be 98; 1, 8, 1, 8, 1, 1, 12, 1, 1, 1, 15, 1, 4, 1, 2, 2, 6, 1, 1, 1, 3, 2, 1, 1, 2, 5, 9, 4, 3, 2, 21, 1, 1, 5, 7, 6, 1, 12, 3, 16, 6, 3, 7, 1, 1, 2, 9, 2, 48, 1, 38, 1, 1, 2, 1, 1, 1, 2, 1, 1, 1, 65, 3, 2, 1, 27, 1, 1, 3, 1, 7, 1, 4, 1, 1, 1, 1, 4, 4, 1, 1, 1, 1, 4, 1, 7, 1, 3, 1, 1, 27, 1, 2, 3, 65, 1, 1, 1, 2, 1, 1, 1, 2, 1, 1, 38, 1, 48, 2, 9, 2, 1, 1, 7, 3, 6, 16, 3, 12, 1, 6, 7, 5, 1, 1, 21, 2, 3, 4, 9, 5, 2, 1, 1, 2, 3, 1, 1, 1, 6, 2, 2, 1, 4, 1, 15, 1, 1, 1, 12, 1, 1, 8, 1, 8, 1, 196, $=u_{157}=2r$.

As 157, the number of quotients in a period, is odd, therefore $x=p_{157}$, $y=q_{157}$ satisfy the equation $x^2-9781y^2=-1$; and $x=p_{314}$, $y=q_{314}$ satisfy the equation $x^2-9781y^2=+1$. It is not necessary to compute the *numerators* of the convergent fractions as $p=rq_m+q_{m-1}$. Computing the values of q_1, q_2, q_3, q_4, etc., we find

$q_{157}=$493415265416374912814471103134560111323246466844777158418926677585 7369909961,

$p_{157}=$487982458985070727523051082184114842770783358477341808754197715936 903165164370.

$x=p_{314}=2p_{157}^2+1=$476253760754232669622915551420643775806417468647845920709133116505163927786611046291325633404816631400075031779842394788655329052356895448229542978234993801,

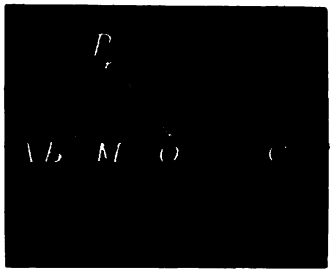
$y=q_{314}=2p_{157}q_{157}=$4815559890373079157588581769809679324712590671132180607164384581211216970331509974781382264086340917459934751261746227674749436227294352561803637330579140.

These prodigious numbers are believed to be the largest of the kind ever computed.

24.—Proposed by James McLaughlin, Mantorville, Dodge County, Minnesota.

An army in the form of a circle, $2a$ miles in diameter, marches due north at the uniform rate of n miles an hour. An officer starts from the rear of the army and rides around it at the uniform rate of m miles an hour, keeping close to the army all the time. Required the equation to the *curve* the officer describes, and the distance he rides while going once around the army.

Solution by E. B. Seitz, Greenville, Darke Co., Ohio; and Artemas Martin, M. A., Erie, Erie Co., Pa.



Let BPC represent the army, B being the rear, O the center, A the initial position of B, P the place of the officer at any time. Draw PM perpendicular to AO, and join O and P.

Put $OP=a$, $\frac{n}{m}=c$, $AM=x$, $PM=y$, $AB=w$, arc $AP=z$.

Then $OM=\sqrt{(a^2-y^2)}$, and $w=x-a\pm\sqrt{(a^2-y^2)}$. (1), the double sign being taken + when P is in the first or fourth quadrant, and — when P is in the second or third quadrant.

We also have $z:w::m:n$, whence $nz=mw$ . . (2), and $dz^2=dx^2+dy^2$ . . (3).

From (1) and (2) we have $z=\frac{m}{n}[x-a\pm\sqrt{(a^2-y^2)}]$ . . . . (4).

Differentiating (4), we have $dz=\frac{m}{n}\left(dx\mp\frac{ydy}{\sqrt{(a^2-y^2)}}\right)$ . . . . (5).

Substituting the value of dz from (5) in (3), reducing, and solving for dx, we find

$$dx=\pm\frac{m^2ydy}{(m^2-n^2)\sqrt{(a^2-y^2)}}\pm\frac{n\sqrt{\left[a^2+\left(\frac{n^2}{m^2-n^2}\right)y^2\right]}dy}{\sqrt{[(m^2-n^2)(a^2-y^2)]}} \quad . . . . (6),$$

the double sign being taken as in (1).

Integrating (6), and observing that when $x=0$, $y=0$, we have for the required equation,

$$x=\frac{m^2}{m^2-n^2}[a\mp\sqrt{(a^2-y^2)}]+\frac{mna}{m^2-n^2}\left[E\left(c,\frac{y}{a}\sqrt{1-c^2}\right)\right] \quad . . . . (7),$$

where $E\left(c,\frac{y}{a}\sqrt{1-c^2}\right)$ represents an elliptic arc, semi-major axis unity, eccentricity c and ordinate $\frac{y}{a}\sqrt{1-c^2}$.

When the officer has gone one-fourth of the way around the army, we have $y=a$, and from (5) and (7) we find that the distance he rides is $z_1=\frac{mna}{m^2-n^2}[1+E(c)]$, where $E(c)$ represents the length of an elliptic quadrant, semi-major axis 1 and eccentricity c. When he has gone half way around the army, $y=0$, and the distance he rides is $z_2=\frac{2mna}{m^2-n^2}[1+E(c)]$. When he has gone three-fourths of the way around, $y=-a$, and the distance he rides is $z_3=\frac{mna}{m^2-n^2}[1+3E(c)]$. When he has gone once around the army, $y=0$, and the distance he rides is $z_4=\frac{4mna}{m^2-n^2}[E(c)]$.

Good solution given by *Henry Heaton*.

25.—Proposed by DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

A perfectly homogeneous sphere has an angular velocity ω about its diameter. If the sphere gradually contract, remaining constantly homogeneous, required the angular velocity when it has half the original diameter.

I.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let r=radius of the sphere before, and r_1=the radius after, contraction and w'=the required angular velocity.

The moment of the momentum about the fixed diameter before contraction was mk^2w, where m is the mass of the sphere and k is the radius of gyration, and after contraction was mk'^2w', and as no forces acted on the sphere during the interval of contraction, $mk^2w=mk'^2w'$. (*Routh's Rigid Dynamics*, 2d edition, Art. 129, p. 195.)

$\therefore\ w'=\frac{mk^2}{mk'^2}w$. But $k^2=\frac{2}{5}r^2$, $k'^2=\frac{2}{5}r_1^2$; $\therefore\ w'=\frac{r^2}{r_1^2}w=4w$ when $r_1=\frac{1}{2}r$.

This solution supposes the mass of the sphere to remain constant during contraction.

Solved also by *Prof. David Trowbridge*, the *Proposer* and *F. P. Matz*.

II.—Solution by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Let R be the radius of the sphere before contraction, r the radius after contraction and ω_1 the angular velocity after contraction.

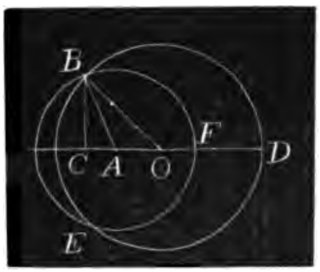
The area described by the radius R in the unit of time is $\frac{1}{2}R^2\omega$, and the area described by the radius r in the same length of time is $\frac{1}{2}r^2\omega_1$. From the principle of the *conservation of areas* we have $\frac{1}{2}r^2\omega_1=\frac{1}{2}R^2\omega$; whence $\omega_1=\frac{R^2}{r^2}\omega,=4\omega$ when $r=\frac{1}{2}R$.

Solved in a similar manner by *Henry Heaton*.

26.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Two equal spheres, radii r, are described within a sphere, radius $2r$; find the average of the volume common to the two spheres.

Solution by HENRY HEATON, B. S., Superintendent of Schools, Sabula, Jackson County, Iowa.



Let O be the center of the sphere whose radius is $2r$, BED a section of a sphere whose radius is r and whose center is O, and A the center of one of the small spheres.

Put x=OA and y=distance between the centers of the small spheres. Let BEF be a section of a sphere whose center is A and whose radius is y.

If $y<(r-x)$ the sphere BEF will lie wholly within BED, and the center of the second small sphere may be on any point of the surface. If $y>(r-x)$ and $<(r+x)$, the sphere BEF will intersect the surface of BED as in the figure, and the center of the second small sphere must lie on that part of the surface of BEF which lies within the surface BED. Draw BC perpendicular to OC, and join BA and BO. Then since $(BO)^2=(AB)^2+(AO)^2+2CA\times AO$ we have $CA=\frac{r^2-x^2-y^2}{2x}$. Hence the surface $BFE=2\pi y\left(y+\frac{r^2-x^2-y^2}{2x}\right)$. A may be on any point of the sphere whose radius is x and whose center is O, x being less than r. The volume common to the two small spheres when the distance between their centers is y is $v=\pi\left(\frac{4}{3}r^3-r^2y+\frac{1}{12}y^3\right)$. The whole number of different positions of the two spheres equals the square of the number of points in the sphere BED Hence the average volume required

$$=\frac{9}{2r^6}\int_0^r\left[\int_0^{r-x}2vy^2dy+\int_{r-x}^{r+x}vy\left(y+\frac{r^2-x^2-y^2}{2x}\right)dy\right]x^2dx,$$

$$=\frac{9\pi}{2r^6}\int_0^r\left(\frac{5}{12}r^6-\frac{1}{4}r^4x^2+\frac{1}{20}r^2x^4-\frac{1}{1260}x^6\right)x^2dx=\frac{136}{315}\pi r^3.$$

Solved also in an elegant manner by the *Proposer*.

27.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

The first of two casks contains a gallons of wine, and the second b gallons of water. Part of the water is poured into the first cask, and then part of the mixture is poured back into the second. Required the probability that not more than $\frac{1}{n}$ of the contents of the second cask is wine.

Solution by HENRY HEATON, B. S., Superintendent of Schools, Sabula, Jackson County, Iowa.

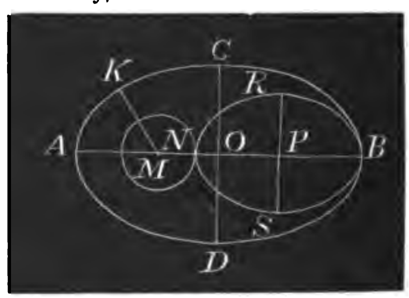
Let x=number of gallons of water poured into the first cask, and y=number of gallons of the mixture poured back into the second. Then the number of gallons in the second cask$=b-x+y$, of which $\frac{ay}{a+x}$ is wine. But $\frac{ay}{a+x}$ must not exceed $\frac{b-x+y}{n}$, hence y must not exceed $\frac{(a+x)(b-x)}{(n-1)a-x}$. Since the cask from which y gallons are taken contains $a+x$ gallons, the probability that y does not exceed $\frac{(a+x)(b-x)}{(n-1)a-x}$ is $\frac{b-x}{(n-1)a-x}$, and that x should have any particular value, $\frac{dx}{b}$. Hence the required probability is

$$p=\frac{1}{b}\int_0^b\frac{(b-x)dx}{(n-1)a-x}=1-\frac{(n-1)a-b}{b}\log\left(\frac{(n-1)a}{(n-1)a-b}\right).$$

28.—Proposed by F. P. MATZ, B. E., B. S., Mathematical Editor *National Educator*, Reading, Berks County, Pennsylvania.

From a point taken at random in the left-hand half of the major axis of an ellipse whose minor axis is unknown, a circle is drawn at random, but so as to lie wholly in the surface of the ellipse. Find the average area of the ellipse whose major axis is that portion of the given major axis between its right-hand extremity and the circumference of the circle.

Solution by E. B. SEITZ, Greenville, Darke County, Ohio.



Let ACBD be the ellipse whose major axis AB is known, and minor axis CD unknown, M the center of the random circle, NRBS the ellipse whose average area is to be determined, RS its minor axis, and MK the normal line to the ellipse.

Put OA$=a$, OC$=w$, AM$=x$, MN$=y$, PR$=z$, NP$=v$, MK$=y_1$, $x_1=\frac{w^2}{a}$ =the radius of curvature at A. Then we have $v=\frac{1}{2}(2a-x-y)$, $y_1=w\sqrt{\left(1-\frac{(a-x)^2}{a^2-w^2}\right)}$, and area NRBS$=\pi vz$. The limits of z are 0 and v; those of y are 0 and x when x is less than x_1, and 0 and y_1 when x is greater than x_1;

those of x are 0 and a; and of w, 0 and a. Hence the required average is

$$J=\frac{\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v \pi vz\,dxdydz+\int_{x_1}^a\int_0^{y_1}\int_0^v \pi vz\,dxdydz\right]dw}{\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dxdydz+\int_{x_1}^a\int_0^{y_1}\int_0^v dxdydz\right]dw}.$$

But $\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v dxdydz+\int_{x_1}^a\int_0^{y_1}\int_0^v dxdydz\right]dw$

$$=\tfrac{1}{2}\int_0^a\left[\int_0^{x_1}\int_0^x(2a-x-y)dxdy+\int_{x_1}^a\int_0^{y_1}(2a-x-y)dxdy\right]dw,$$

$$=\tfrac{1}{4}\int_0^a\left[\int_0^{x_1}(4ax-3x^2)dx+\int_{x_1}^a[(2a-x)^2-(2a-x-y_1)^2]dx\right]dw,$$

$$=\frac{1}{12}\int_0^a\left[3aw(a^2-w^2)^{\frac{1}{2}}\cos^{-1}\left(\frac{w}{a}\right)+2a^2w+aw^2-2w^3+\frac{2w^4}{a}\right]dw=\frac{1}{360}a^4(15\pi+17).$$

$$\therefore\ J=\frac{360}{a^4(15\pi+17)}\int_0^a\left[\int_0^{x_1}\int_0^x\int_0^v \pi vz\,dxdydz+\int_{x_1}^a\int_0^{y_1}\int_0^v \pi vz\,dxdydz\right]dw,$$

$$=\frac{45\pi}{2a^4(15\pi+17)}\int_0^a\left[\int_0^{x_1}\int_0^x(2a-x-y)^3dxdy+\int_{x_1}^a\int_0^{y_1}(2a-x-y)^3dxdy\right]dw,$$

$$=\frac{45\pi}{8a^4(15\pi+17)}\int_0^a\left[\int_0^{x_1}(32a^3x-72a^2x^2+56ax^3-15x^4)dx\right.$$
$$\left.+\int_{x_1}^a[(2a-x)^4-(2a-x-y_1)^4]dx\right]dw,$$

$$=\frac{3\pi}{16a^4(15\pi+17)}\int_0^a\left[105a^3w(a^2-w^2)^{\frac{1}{2}}\cos^{-1}\left(\frac{w}{a}\right)+136a^4w-129a^3w^2-128a^2w^3\right.$$
$$\left.+295aw^4-8w^5-\frac{92w^6}{a}+\frac{16w^8}{a^3}\right]dw,\qquad=\frac{\pi a^2}{672}\left(\frac{2205\pi+2012}{15\pi+17}\right).$$

29.—Proposed by E. B. Seitz, Greenville, Darke County, Ohio.

Two points are taken at random in the surface of a given circle, but on opposite sides of a given diameter. Find (1) the chance that the distance between the points is less than the radius of the circle, and (2) the average distance between them.

Solution by the Proposer.

1.—Let AB be the given diameter, OD the radius perpendicular to AB. With A as a center describe the arc OC, and from any point M in the surface AOC and with a radius equal to AO describe the arc PN. If M is one of the random points, and the second point Q be taken anywhere in the surface APSN, the distance between them will be less than the radius of the circle. Put OA$=1$, OM$=x$, $\angle$AOM$=\theta$, $\angle$ONM$=\varphi$, $\angle$MOP$=\psi$, area APSN$=u$. Then $x=\operatorname{cosec}\theta\sin\varphi=2\cos\psi$,

$dx=\operatorname{cosec}\theta\cos\varphi d\varphi=-2\sin\psi d\psi$, and

$u=$POR$+$PMS$-$POM$+$NMS$-$AOR$-$MON,

$$=\psi-\sin\psi\cos\psi-\tfrac{1}{2}\varphi-\tfrac{1}{2}\sin\varphi\cos\varphi+\tfrac{1}{2}\cot\theta\sin^2\varphi.$$

When M is anywhere in the surface ODC, the number of favorable positions of Q is $u_1 = \frac{1}{2}\pi - \varphi - \sin\varphi\cos\varphi$. Hence the required chance is

$$p = \frac{\int_0^{\frac{1}{3}\pi}\int_0^1 uxdxd\theta + \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left[\int_0^{2\cos\theta} uxdx + \int_{2\cos\theta}^1 u_1\, xdx\right]d\theta}{\int_0^{\frac{1}{2}\pi}\int_0^1 \frac{1}{2}\pi xdxd\theta},$$

$$= \frac{8}{\pi^2}\int_0^{\frac{1}{3}\pi}\int_0^1 uxdxd\theta + \frac{8}{\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left[\int_0^{2\cos\theta} uxdx + \int_{2\cos\theta}^1 u_1\, xdx\right]d\theta.$$

But when $x=0$, $\varphi=0$ and $\psi=\frac{1}{2}\pi$; when $x=1$, $\varphi=\theta$ and $\psi=\frac{1}{3}\pi$; and when $x=2\cos\theta$, $\varphi=\pi-2\theta$ and $\psi=\theta$.

$$\therefore\ p = \frac{8}{\pi^2}\int_0^{\frac{1}{3}\pi}\left[\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} 4(\psi - \sin\psi\cos\psi)\sin\psi\cos\psi d\psi - \int_0^{\theta}\tfrac{1}{2}(\varphi + \sin\varphi\cos\varphi - \cot\theta\sin^2\varphi)\operatorname{cosec}^2\theta\sin\varphi\cos\varphi d\varphi\right]d\theta$$

$$+ \frac{8}{\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\left[\int_{\theta}^{\frac{1}{2}\pi} 4(\psi - \sin\psi\cos\psi)\sin\psi\cos\psi d\psi - \int_0^{\pi-2\theta}\tfrac{1}{2}(\varphi + \sin\varphi\cos\varphi - \cot\theta\sin^2\varphi)\operatorname{cosec}^2\theta\sin\varphi\cos\varphi d\varphi\right.$$

$$\left. + \int_{\pi-2\theta}^{\theta}(\tfrac{1}{2}\pi - \varphi - \sin\varphi\cos\varphi)\operatorname{cosec}^2\theta\sin\varphi\cos\varphi d\varphi\right]d\theta,$$

$$= \frac{1}{2\pi^2}\int_0^{\frac{1}{3}\pi}(4\pi - 4\theta + \theta\operatorname{cosec}^2\theta - \cot\theta - 3\sqrt{3})d\theta$$

$$+ \frac{1}{2\pi^2}\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}(8\pi - 16\theta - \pi\operatorname{cosec}^2\theta + 4\theta\operatorname{cosec}^2\theta - 4\cot\theta - 8\sin\theta\cos\theta)d\theta, = \tfrac{4}{3} - \frac{\sqrt{3}}{2\pi}.$$

2.—Let M be any point in the semicircle ADB, and Q any point in the semicircle APB. Put $OA=r$, $OM=x$, $OQ=y$, $MQ=z$, $\angle AOM=\theta$, $\angle AOQ=\varphi$. Then $z=[x^2+y^2-2xy\cos(\theta+\varphi)]^{\frac{1}{2}}$, and the average distance between the points is

$$\Delta = \frac{\int_0^{\pi}\int_0^{\pi}\int_0^{r}\int_0^{x} zd\theta d\varphi xdxydy}{\int_0^{\pi}\int_0^{\pi}\int_0^{r}\int_0^{x} d\theta d\varphi xdxydy}, = \frac{8}{\pi^2 r^4}\int_0^{\pi}\int_0^{\pi}\int_0^{r}\int_0^{x}[x^2+y^2-2xy\cos(\theta+\varphi)]^{\frac{1}{2}}d\theta d\varphi xdxydy,$$

$$= \frac{2}{3\pi^2 r^4}\int^{\pi}\int_0^{\pi}\int_0^{r}\Big[8\sin\tfrac{1}{2}(\theta+\varphi) + 40\sin\tfrac{1}{2}(\theta+\varphi)\cos^2\tfrac{1}{2}(\theta+\varphi) - 48\sin\tfrac{1}{2}(\theta+\varphi)\cos^4(\theta+\varphi)$$

$$+ 3\cos2(\theta+\varphi) - 1 + 6\sin^2(\theta+\varphi)\cos(\theta+\varphi)\log\left(\frac{1+\sin\frac{1}{2}(\theta+\varphi)}{\sin\frac{1}{2}(\theta+\varphi)}\right)\Big]d\theta d\varphi x^4dx,$$

$$= \frac{2r}{15\pi^2}\int_0^{\pi}\int_0^{\pi}\Big[8\sin\tfrac{1}{2}(\theta+\varphi) + 40\sin\tfrac{1}{2}(\theta+\varphi)\cos^2\tfrac{1}{2}(\theta+\varphi) - 48\sin\tfrac{1}{2}(\theta+\varphi)\cos^4\tfrac{1}{2}(\theta+\varphi)$$

$$+ 3\cos2(\theta+\varphi) - 1 + 6\sin^2(\theta+\varphi)\cos(\theta+\varphi)\log\left(\frac{1+\sin\frac{1}{2}(\theta+\varphi)}{\sin\frac{1}{2}(\theta+\varphi)}\right)\Big]d\theta d\varphi,$$

$$= \frac{4r}{45\pi^2}\int_0^{\pi}[32\sin\tfrac{1}{2}\theta + 32\cos\tfrac{1}{2}\theta + 16\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta + 16\sin\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta$$

$$- 24\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 24\sin\tfrac{1}{2}\theta\cos^4\tfrac{1}{2}\theta + 3\sin^3\theta\log\tan\tfrac{1}{4}\theta + 3\sin^3\theta\log\tan\tfrac{1}{4}(\pi-\theta)]d\theta,$$

$$= \frac{1472r}{135\pi^2}.$$

30.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

A straight tree growing vertically on the side of a mountain was broken by the wind, but not severed; find the chance that the top reaches to the ground.

I.—Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Let OA represent the tree, AOM a vertical plane perpendicular to the mountain side, and intersecting it in OM, and AON any vertical plane intersecting the mountain side in ON. Take O as the center of a sphere, and the intersection of its surface with the planes AOM, AON and MON will form a right-angled spherical triangle BCD. When the wind blows in the direction of the plane AON, let H be the point at which the tree must break, that the top A may just reach the ground at K.

Put OH$=x$, OA$=a$, $\angle$AOM$=\frac{1}{2}\pi-\theta$, the complement of the elevation of the mountain, diedral MOAN$=\varphi$, $\angle$AON$=\psi$. Then in the spherical triangle BCD we have $\tan\psi=\cot\theta\sec\varphi$, whence $\sin\psi=\dfrac{\cot\theta\sec\varphi}{\sqrt{(1+\cot^2\theta\sec^2\varphi)}}$.

In the right-angled triangle HKO we have HK$=$OH$\sin\psi$, or $a-x=x\sin\psi$, whence $x=\dfrac{a}{1+\sin\psi}$. Substituting the value of $\sin\psi$, we have

$$x=a[1+\cot^2\theta\sec^2\varphi-\cot\theta\sec\varphi\sqrt{(1+\cot^2\theta\sec^2\varphi)}].$$

When $\varphi<\frac{1}{2}\pi$, the required chance for a given direction of the wind, is $\dfrac{x}{a}$; but when $\varphi>\frac{1}{2}\pi$, the required chance is $\frac{1}{2}$. Hence, integrating first with respect to θ, we find for the chance required

$$p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{x}{a}d\varphi d\theta+\int_{\frac{1}{2}\pi}^{\pi}\int_0^{\frac{1}{2}\pi}\frac{1}{2}d\varphi d\theta}{\int_0^{\pi}\int_0^{\frac{1}{2}\pi}d\varphi d\theta},=\frac{1}{4}+\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}[1+\cot^2\theta\sec^2\varphi$$

$$-\cot\theta\sec\varphi\sqrt{(1+\cot^2\theta\sec^2\varphi)}]d\varphi d\theta,$$

$$=\frac{1}{4}+\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\Big[\theta-\theta\sec^2\varphi-\cot\theta\sec^2\varphi+\sec\varphi\sqrt{(1+\cot^2\theta\sec^2\varphi)}$$

$$+\tan\varphi\sec\varphi\cot^{-1}(\cot\varphi\sqrt{1+\cot^2\theta\sec^2\varphi})\Big]_0^{\frac{1}{2}\pi}d\varphi,$$

$$=\frac{1}{4}+\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}[\tfrac{1}{2}\pi(1-\sec^2\varphi)+\varphi\tan\varphi\sec\varphi+\sec\varphi]d\varphi,$$

$$=\frac{1}{4}+\frac{2}{\pi^2}\Big[\tfrac{1}{2}\pi(\varphi-\tan\varphi)+\varphi\sec\varphi\Big]_0^{\frac{1}{2}\pi}=\frac{3}{4}-\frac{2}{\pi^2}.$$

II.—Solution by Henry Heaton, B. S., Superintendent of Schools, Sabula, Jackson County, Iowa.

In the figure ACD is the side of the mountain, AB the stump of the tree, ABC the plane passing through the tree perpendicular to ACD, BAD the plane in which the tree is supposed to fall, BCD the horizontal plane through B, and BE the perpendicular from B upon AD. Put $a=$hight of tree, BE$=x$, $\angle$BAC$=\beta$ and $\angle$CBD$=\theta$. If the length of the top$=x$, AB$=a-x$,

$$\text{BD}=\frac{x(a-x)}{\sqrt{(a^2-2ax)}},\ \text{BC}=(a-x)\tan\beta\text{ and BD}=\frac{(a-x)\tan\beta}{\cos\theta};$$

$$\therefore\ \frac{x}{\sqrt{(a^2-2ax)}}=\frac{\tan\beta}{\cos\theta},\ \text{and}\ x=\frac{a\tan\beta}{\cos^2\theta}[\sqrt{(\sec^2\beta-\sin^2\theta)}-\tan\beta].$$

If $\theta<\frac{1}{2}\pi$ and the top shorter than x, or if $\theta>\frac{1}{2}\pi$ and $<\pi$ and the top shorter than $\frac{1}{2}a$, it can not reach the ground. Hence the required probability is

$$p=\frac{3}{4}-\frac{\tan\beta}{\pi}\int_0^{\frac{1}{2}\pi}\left[\sqrt{(\sec^2\beta-\sin^2\theta)}-\tan\beta\right]\frac{d\theta}{\cos^2\theta},$$

$$=\frac{3}{4}-\frac{\tan\beta}{\pi}\left[\tan\theta\{\sqrt{\sec^2\beta-\sin^2\theta)}-\tan\beta\}\right]_0^{\frac{1}{2}\pi}-\frac{\tan\beta\sec\beta}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\beta\sin^2\theta d\theta}{\sqrt{(1-\cos^2\beta\sin^2\theta)}},$$

$$=\frac{3}{4}-\frac{\tan\beta\sec\beta}{\pi}\left[F(\cos\beta,\tfrac{1}{2}\pi)-E(\cos\beta,\tfrac{1}{2}\pi)\right].$$

If β is supposed to be unknown, the required probability is

$$p=\frac{3}{4}-\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{\sin\beta\sin^2\theta d\theta d\beta}{\sqrt{(1-\cos^2\beta\sin^2\theta)}}=\frac{3}{4}-\frac{2}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin\theta d\theta=\frac{3}{4}-\frac{2}{\pi^2}.$$

Chas. H. Kummell gave an excellent solution on the supposition that the elevation of the mountain is known.

31.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

A sphere, radius r, and a candle are placed at random on a round table, radius R, the hight of the candle being equal to the radius of the sphere Find the average surface of the sphere illuminated by the candle.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Let BCED be the table, A the point at which the sphere touches the table, M the place of the candle, O the the center of the table. With A as a center describe the arc DKE through M.

Put AM=x, OA=y, OD=R, $\angle$OAD=θ, $\angle$AOD=φ, $\angle$ADO=ψ, u=the area of the surface of the sphere illuminated. Then $\cos\theta=\dfrac{x^2+y^2-R^2}{2xy}$ . . . . . (1),

$\cos\varphi=\dfrac{R^2+y^2-x^2}{2Ry}$ . . . . (2), $\cos\psi=\dfrac{R^2+x^2-y^2}{2Rx}$ . . . (3),

$y\sin\theta=R\sin\psi$ . . . . (4), $y\cos\varphi=R-x\cos\psi$ . . . . (5), $u=2\pi r^2\left(1-\dfrac{r}{x}\right)$.

Now if we regard x a constant, and less than R, A may move over the surface of the circle, center O and radius $R-x$, and for each position of A, M may move over the circumference of the circle, center A and radius x, and the amount of the surface illuminated will remain the same. But when $x<R$ and $y>R-x$, or when $x>R$ and $y>x-R$, for each position of A, M may move over the arc DKE, and the amount of surface illuminated will remain the same. Hence the required average is

$$S=\frac{\int_r^R\int_0^{R-x}u\,.\,2\pi xdx\,.\,2\pi ydy+\int_r^R\int_{R-x}^R u\,.\,2\theta xdx\,.\,2\pi ydy+\int_R^{2R}\int_{x-R}^R u\,.\,2\theta xdx\,.\,2\pi ydy}{\int_r^R\int_0^{R-x}2\pi xdx\,.\,2\pi ydy+\int_r^R\int_{R-x}^R 2\theta xdx\,.\,2\pi ydy+\int_R^{2R}\int_{x-R}^R 2\theta xdx\,.\,2\pi ydy}.$$

But $\int 2\theta ydy=\theta y^2-\int y^2d\theta$. From (1) and (2), $xy\sin\theta d\theta=-R\cos\varphi dy$ . . . . . (6).

From (4) and (6), $x\sin\psi d\theta=-\cos\varphi dy$ . . . (7). From (3), $Rx\sin\psi d\psi=ydy$ . . (8).

From (7) and (8), $yd\theta=-R\cos\varphi d\psi$ . . . (9), and from (5) and (9),

$$y^2d\theta=-R(R-x\cos\psi)d\psi. \quad \therefore \quad \int 2\theta ydy=\theta y^2+R\psi-Rx\sin\psi.$$

Hence, observing that when $y=R-x$, $\theta=\pi$ and $\psi=0$; when $y=x-R$, $\theta=\psi=0$, and when $y=R$, $\theta=\psi=\cos^{-1}\left(\frac{x}{2R}\right)$, we have

$$S=\frac{\int_r^{2R}\left[4R^2\cos^{-1}\left(\frac{x}{2R}\right)-x\sqrt{(4R^2-x^2)}\right]uxdx}{\int_r^{2R}\left[4R^2\cos^{-1}\left(\frac{x}{2R}\right)-x\sqrt{(4R^2-x^2)}\right]xdx},$$

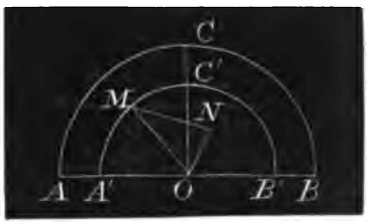
$$=2\pi r^2-\frac{2\pi r^3\int_r^{2R}\left[4R^2\cos^{-1}\left(\frac{x}{2R}\right)-x\sqrt{(4R^2-x^2)}\right]dx}{\int_r^{2R}\left[4R^2\cos^{-1}\left(\frac{x}{2R}\right)-x\sqrt{(4R^2-x^2)}\right]xdx},$$

$$=2\pi r^2\left[\frac{24R^2(R^2+r^2)\cos^{-1}\left(\frac{r}{2R}\right)-r(26R^2+r^2)\sqrt{(4R^2-r^2)}}{24R^2(R^2-r^2)\cos^{-1}\left(\frac{r}{2R}\right)+3r(2R^2+r^2)\sqrt{(4R^2-r^2)}}\right].$$

32.—Proposed by E. B. Seitz, Greenville, Darke County, Ohio.

A radius is drawn dividing a given semicircle into two quadrants, and a point taken at random in each quadrant; find the average distance between them.

Solution by the Proposer.



Let ACB be the given semicircle, O its center, AOC and BOC the two quadrants, M any point in the quadrant AOC, A′C′B′ the concentric semicircle through M, and N any point in the quadrant B′OC′. It is evidently necessary to consider only those positions of the two points in which N is confined to the quadrant B′OC′.

Put OM$=x$, ON$=y$, OA$=r$, ∠COM$=\theta$, ∠CON$=\varphi$. Then MN$=[x^2+y^2-2xy\cos(\theta+\varphi)]^{\frac{1}{2}}$; the limits of y are 0 and x; of x, 0 and r; of φ, 0 and $\frac{1}{2}\pi$; and of θ, 0 and $\frac{1}{2}\pi$. Hence the required average is

$$\Delta=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^x\left[x^2+y^2-2xy\cos(\theta+\varphi)\right]^{\frac{1}{2}}d\theta d\varphi xdxydy}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^x d\theta d\varphi xdxydy},$$

$$=\frac{32}{\pi^2r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^r\int_0^x[x^2+y^2-2xy\cos(\theta+\varphi)]^{\frac{1}{2}}d\theta d\varphi xdxydy,$$

$$=\frac{2}{3\pi^2r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^r\Big[8\sin\tfrac{1}{2}(\theta+\varphi)+40\sin\tfrac{1}{2}(\theta+\varphi)\cos^2\tfrac{1}{2}(\theta+\varphi)-48\sin\tfrac{1}{2}(\theta+\varphi)\cos^4(\theta+\varphi)$$

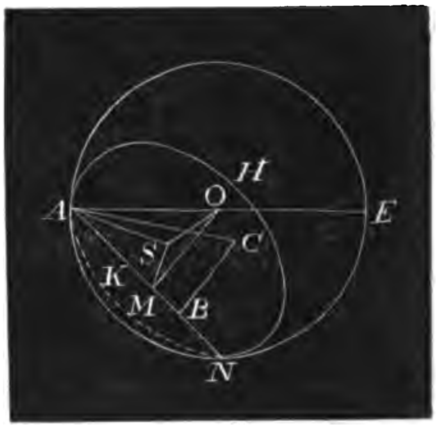
$$+3\cos2(\theta+\varphi)-1+6\sin^2(\theta+\varphi)\cos(\theta+\varphi)\log\left(\frac{1+\sin\frac{1}{2}(\theta+\varphi)}{\sin\frac{1}{2}(\theta+\varphi)}\right)\Big]d\theta d\varphi x^4dx,$$

$$=\frac{16r}{15\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\Big[16\sin^3\tfrac{1}{2}(\theta+\varphi)-2+3\cos^2(\theta+\varphi)+12\sin^3\tfrac{1}{2}(\theta+\varphi)\cos(\theta+\varphi)$$
$$+3\sin^2(\theta+\varphi)\cos(\theta+\varphi)\log\left(\frac{1+\sin\frac{1}{2}(\theta+\varphi)}{\sin\frac{1}{2}(\theta+\varphi)}\right)\Big]d\theta d\varphi,$$
$$=\frac{16r}{45\pi^2}\int_0^{\frac{1}{2}\pi}\Big[24\cos\tfrac{1}{2}\theta+32\cos^3\tfrac{1}{2}\theta-24\cos^5\tfrac{1}{2}\theta-3\sin2\theta-24\cos\tfrac{1}{2}(\tfrac{1}{2}\pi+\theta)-32\cos^3\tfrac{1}{2}(\tfrac{1}{2}\pi+\theta)$$
$$+24\cos^5\tfrac{1}{2}(\tfrac{1}{2}\pi+\theta)-3\sin^3\theta\log\left(\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)+3\cos^3\theta\log\left(\frac{1+\sin\frac{1}{2}(\frac{1}{2}\pi+\theta)}{\sin\frac{1}{2}(\frac{1}{2}\pi+\theta)}\right)\Big]d\theta,$$
$$=\frac{32r}{135\pi^2}\left[94\sqrt{2}-95+6\log\left(\frac{1+\sqrt{2}}{2}\right)\right].$$

33.—Proposed by E. B. Seitz, Greenville, Darke County, Ohio.

Three points being taken at random within a sphere, find the average area of the triangle formed by joining them.

Solution by the Proposer.



Let Δ be the required average, and Δ_1 the average area of the triangle when one of the points is fixed on the surface of the sphere. Then if x is the variable distance of one of the points from the center of the sphere, the other two points being confined to the concentric sphere whose radius is x, we have

$$\Delta=\frac{\int_0^r\frac{x^2}{r^2}\Delta_1\left(\frac{4}{3}\pi x^3\right)^2.4\pi x^2dx}{\int_0^r\left(\frac{4}{3}\pi x^3\right)^2.4\pi x^2dx}=\frac{9}{11}\Delta_1.$$

Let O be the center of the sphere, A the fixed point on the surface, B, C the two random points within the sphere, ANEF the great circle through B, AKNH the small circle through B, C, S its center, M the middle point of AN. Put AB$=x$, AC$=y$, AS$=z$, AO$=r$, $\angle$BAO$=\theta$, $\angle$OMS$=\varphi$, $\angle$CAB$=\psi$, $\angle$SAM$=\omega$. Then AM$=r\cos\theta=z\cos\omega$, SM$=r\sin\theta\cos\varphi=z\sin\omega$,

$z=r(\cos^2\theta+\sin^2\theta\cos^2\varphi)^{\frac{1}{2}}$, area ABC$=\frac{1}{2}xy\sin\psi$.

An element of the sphere at C is $y^2\sin\psi\, dy d\psi d\varphi\,.\,2\pi x^2\sin\theta\, dx d\theta$; the limits of y are 0 and $2z\cos(\psi-\omega)=z_1$; of ψ, $\omega-\frac{1}{2}\pi=\omega_1$ and $\omega+\frac{1}{2}\pi=\omega_2$; of x, 0 and $2r\cos\theta=x_1$; of θ, 0 and $\frac{1}{2}\pi$; and of φ, 0 and π. Hence, since the whole number of ways the two points can be taken within the sphere is $\frac{16}{9}\pi^2r^6$, we have

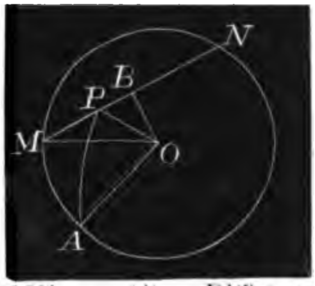
$$\Delta_1=\frac{9}{16\pi^2r^6}\int_0^\pi\int_0^{\frac{1}{2}\pi}\int_0^{x_1}\int_{\omega_1}^{\omega_2}\int_0^{z_1}\tfrac{1}{2}xy^3\sin^2\psi d\varphi\sin\theta d\theta\,.\,2\pi x^2dxd\psi dy,$$
$$=\frac{9}{4\pi r^6}\int_0^\pi\int_0^{\frac{1}{2}\pi}\int_0^{x_1}\int_{\omega_1}^{\omega_2}z^4\cos^4(\psi-\omega)\sin^2\psi d\varphi\sin\theta d\theta x^3dxd\psi,$$
$$=\frac{9}{64r^2}\int_0^\pi\int_0^{\frac{1}{2}\pi}\int_0^{x_1}(\cos^4\theta+6\sin^2\theta\cos^2\theta\cos^2\varphi+5\sin^4\theta\cos^4\varphi)d\varphi\sin\theta d\theta x^3dx,$$
$$=\frac{9r^2}{16}\int_0^\pi\int_0^{\frac{1}{2}\pi}(5\cos^4\varphi+6\cos^2\varphi\cos^2\theta-10\cos^4\varphi\cos^2\theta-6\cos^2\varphi\cos^4\theta$$
$$+5\cos^4\varphi\cos^4\theta+\cos^4\theta)d\varphi\sin\theta\cos^4\theta d\theta,$$

$= \frac{r^2}{112}\int_0^\pi (16+10\cos 2\varphi+\cos 4\varphi)d\varphi = \frac{1}{7}\pi r^2$. $\therefore\ \Delta = \frac{9}{11}\Delta_1 = \frac{9}{77}\pi r^2 = \frac{9}{77}$ of the area of a great circle.

34.—Prize Problem. Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

A boy stepped upon a horizontal turn-table, while it was in motion, and walked across it, keeping all the time in the same vertical plane. The boy's velocity is supposed to be uniform in his track on the table, and the motion of the table towards him. The velocity of a point in the circumference of the turn-table is n times the velocity of the boy along the curve he describes Required—the nature of the *curve* the boy describes on the table, and the distance he walks while crossing it (1) when n is less than 1, (2) when $n=1$ and (3) when n is greater than 1.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.



Let AMN represent the turn-table, and let MN be the intersection of the fixed vertical plane with the table, O the center of the table, A the point in the circumference at which the boy stepped upon the table, P his position at any time while crossing it, AP the corresponding arc described by the boy, OB the perpendicular on MN, the motion of the table being from N towards M.

Put $OM=a$, $OP=r$, $OB=b$, $BM=c$, arc $AP=z$, $\angle AOP=\theta$, $\angle AOM=\varphi$, $\angle OPN=\mu$, $\angle OMN=\beta$. Then arc $AM=n\,.$ arc AP, or $a\varphi=nz$ . . . (1), $\varphi=\theta-\mu+\beta$ . . . . (2), $r=b\operatorname{cosec}\mu$ . . . (3), $dz^2=dr^2+r^2d\theta^2$ . . . . (4). Differentiating (1), (2) and (3), we find $d\varphi=\frac{n}{a}dz$ . . . . . . (5), $d\varphi=d\theta-d\mu$ . . . . . . (6), $dr=-b\cos\mu\operatorname{cosec}^2\mu d\mu$ . . . . . . (7). From (5) and (6), $dz=\frac{a}{n}(d\theta-d\mu)$ . . . (8).

Substituting the values of r, dr and dz from (3), (7) and (8) in (4), and solving for $d\theta$, we find

$$d\theta=\frac{a^2\sin^2\mu d\mu}{a^2\sin^2\mu-n^2b^2}\pm\frac{nbd\mu}{\sin\mu(a^2\sin^2\mu-n^2b^2)}\sqrt{[a^2-(a^2+n^2b^2)\cos^2\mu]}\quad . \ . \ . \quad (9).$$

I.—When $n<1$, by integrating (9), taking the double sign $+$, we have

$$\theta=\mu-\frac{nb}{2(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{(a^2-n^2b^2)^{\frac{1}{2}}+nb\cot\mu}{(a^2-n^2b^2)^{\frac{1}{2}}-nb\cot\mu}\right)+nb\int\frac{\sqrt{[a^2-(a^2+n^2b^2)\cos^2\mu]}\,d\mu}{\sin\mu(a^2\sin^2\mu-n^2b^2)}\quad (10).$$

Put $a\cos\psi=(a^2+n^2b^2)^{\frac{1}{2}}\cos\mu$ . . . . (11). Then we have

$$\int\frac{\sqrt{[a^2-(a^2+n^2b^2)\cos^2\mu]}d\mu}{\sin\mu(a^2\sin^2\mu-n^2b^2)}=[a(a^2+n^2b^2)^{\frac{1}{2}}]^3\int\frac{\sin^2\psi d\psi}{(a^2\sin^2\psi+n^2b^2)(a^4\sin^2\psi-n^4b^4)}$$

$$=\cos^{-1}\left(\frac{n^2b^2\cos^2\psi}{a^2\sin^2\psi+n^2b^2}\right)^{\frac{1}{2}}-\frac{nb}{2(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{(a^4-n^4b^4)^{\frac{1}{2}}+n^2b^2\cot\psi}{(a^4-n^4b^4)^{\frac{1}{2}}-n^2b^2\cot\psi}\right).$$

Substituting in (10), then substituting the values of the functions of μ and ψ, determined from (13) and (11), and observing that $\theta=0$ when $r=a$ and $\mu=\beta$, we have for the polar equation to the curve

$$\theta=\sin^{-1}\left(\frac{b}{r}\right)-\sin^{-1}\left(\frac{b}{a}\right)+\cos^{-1}\left(\pm\frac{n}{a}(r^2-b^2)^{\frac{1}{2}}\right)-\cos^{-1}\left(\frac{nc}{a}\right)$$

$$-\frac{nb}{(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{[(a^2-n^2b^2)^{\frac{1}{2}}(a^2-n^2c^2)^{\frac{1}{2}}-n^2bc][(a^2-n^2b^2)^{\frac{1}{2}}\pm n(r^2-b^2)^{\frac{1}{2}}]}{[(a^2-n^2b^2)^{\frac{1}{2}}+nc][(a^2-n^2b^2)^{\frac{1}{2}}(a^2+n^2b^2-n^2r^2)^{\frac{1}{2}}\mp n^2b(r^2-b^2)^{\frac{1}{2}}]}\right)\ .\ (12),$$

the sign $\pm$ being taken $+$ when $\mu<\frac{1}{2}\pi$, and $-$ when $\mu>\frac{1}{2}\pi$; and the sign $\mp$ being taken $-$ when $\mu<\frac{1}{2}\pi$, and $+$ when $\mu>\frac{1}{2}\pi$.

When $r=a$ and $\mu=\pi-\beta$, from (1), (2) and (12) we we find

$$z_1=\frac{2a}{n}\sin^{-1}\left(\frac{nc}{a}\right)+\frac{2ab}{(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{a(a^2-n^2b^2)^{\frac{1}{2}}+nac}{(a^2-n^2b^2)^{\frac{1}{2}}(a^2-n^2c^2)^{\frac{1}{2}}-n^2bc}\right)\ldots(13),$$

the distance the boy walks while crossing the table.

If $b=0$, (12) becomes $\theta=\cos^{-1}\left(\pm\frac{nr}{a}\right)-\cos^{-1}(n)$, which is the equation to a circle, radius $\frac{a}{2n}$, passing through O and A; hence if the boy walks across the center of the table, he will describe the arc of a circle, whose length is $\frac{2a}{n}\sin^{-1}(n)$.

II.—When $n=1$, the logarithmic part of (12) becomes infinite, which shows that it is impossible for the boy to cross the table in a plane that cuts the table on the side from which it is moving. But if the vertical plane cuts the table on the side toward which it is moving, we must substitute $-b$ for b, in (12), and then making $n=1$ we find

$$\theta=\cos^{-1}\left(\pm\frac{(r^2-b^2)^{\frac{1}{2}}}{a}\right)-\sin^{-1}\left(\frac{b}{r}\right)+\frac{b}{c}\log\left(\frac{bc\pm b(r^2-b^2)^{\frac{1}{2}}}{c(a^2+b^2-r^2)^{\frac{1}{2}}\pm b(r^2-b^2)^{\frac{1}{2}}}\right)\quad(14)$$

Substituting $-b$ for b, and making $n=1$, in (23), we have

$$z_1=2a\sin^{-1}\left(\frac{c}{a}\right)+\frac{2ab}{c}\log\left(\frac{b}{a}\right),$$ the distance the boy walks.

If $b=0$, (14) becomes $\theta=\cos^{-1}\left(\pm\frac{r}{a}\right)$, which is the equation to a circle on OA as diameter; hence if the boy walks across the center of the table, he will describe the circumference of a circle, and will leave the table at the same point at which he stepped upon it

III—When $n>1$, the vertical plane must cut the table on the side toward which it is moving, in order that the boy may cross the table. Hence, if we substitute $-b$ for b and $-\mu$ for μ in (9), and take the double sign $-$, we have

$$d\theta=\frac{a^2\sin^2\mu d\mu}{a^2\sin^2\mu-n^2b^2}-\frac{nbd\mu}{\sin\mu(a^2\sin^2\mu-n^2b^2)}\sqrt{[a^2-(a^2+n^2b^2)\cos^2\mu]}\quad\ldots(15).$$

1. When β is not greater than 45°, or when β is not less than 45° and $n<\frac{a}{b}$. we have by integrating (15) for the equation to the curve

$$\theta=\sin^{-1}\left(\frac{b}{a}\right)-\sin^{-1}\left(\frac{b}{r}\right)+\cos^{-1}\left(\pm\frac{n}{a}(r^2-b^2)^{\frac{1}{2}}\right)-\cos^{-1}\left(\frac{nc}{a}\right)$$
$$+\frac{nb}{(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{[n^2bc+(a^2-n^2b^2)^{\frac{1}{2}}(a^2-n^2c^2)^{\frac{1}{2}}]\,[n(r^2-b^2)^{\frac{1}{2}}\pm(a^2-n^2b^2)^{\frac{1}{2}}]}{[nc+(a^2-n^2b^2)^{\frac{1}{2}}]\,[n^2b(r^2-b^2)^{\frac{1}{2}}\pm(a^2-n^2b^2)^{\frac{1}{2}}(a^2+n^2b^2-n^2r^2)^{\frac{1}{2}}]}\right)\quad.(16).$$

If $\beta<45°$, n can not be greater than $\frac{a}{c}$. When $r=a$ and $\mu=\pi-\beta$, from (1), (2) and (16) we find $z_1=\frac{2a}{n}\sin^{-1}\left(\frac{c}{a}\right)+\frac{2ab}{(a^2-n^2b^2)^{\frac{1}{2}}}\log\left(\frac{n^2bc+(a^2-n^2b^2)^{\frac{1}{2}}(a^2-n^2c^2)^{\frac{1}{2}}}{nac+a(a^2-n^2b^2)^{\frac{1}{2}}}\right)$ (17), the distance the boy walks.

2. When β is not less than 45° and $n>\frac{a}{b}$, we find by integrating (15) for the

equation to the curve

$$\theta=\sin^{-1}\left(\frac{b}{a}\right)-\sin^{-1}\left(\frac{b}{r}\right)+\cos^{-1}\left(\pm\frac{n}{a}(r^2-b^2)^{1/2}\right)-\cos^{-1}\left(\frac{nc}{a}\right)$$
$$+\frac{nb}{(n^2b^2-a^2)^{1/2}}\cot^{-1}\left(\pm\frac{n^3b(r^2-b^2)+(n^2b^2-a^2)(a^2+n^2b^2-n^2r^2)^{1/2}}{n(n^2b^2-a^2)^{1/2}(r^2-b^2)^{1/2}[nb-(a^2+n^2b^2-n^2r^2)^{1/2}]}\right)$$
$$-\frac{nb}{(n^2b^2-a^2)^{1/2}}\cot^{-1}\left(\frac{n^3bc^2+(n^2b^2-a^2)(a^2-n^2c^2)^{1/2}}{nc(n^2b^2-a^2)^{1/2}[nb-(a^2-n^2c^2)^{1/2}]}\right)\quad . \quad . \quad . \quad . \quad (18),$$

in which n can not be greater than $\frac{a}{c}$. From (1), (2) and (18), when $r=a$ and $\mu=\pi-\beta$, we find

$$z_1=\frac{2a}{n}\sin^{-1}\left(\frac{nc}{a}\right)$$
$$+\frac{\pi ab}{(n^2b^2-a^2)^{1/2}}-\frac{2ab}{(n^2b^2-a^2)^{1/2}}\cot^{-1}\left(\frac{n^3bc^2+(n^2b^2-a^2)(a^2-n^2c^2)^{1/2}}{nc(n^2b^2-a^2)^{1/2}[nb-(a^2-n^2c^2)^{1/2}]}\right)(19),$$

the distance the boy walks.

3. When β is not less than 45° and $n=\frac{a}{b}$, (15) becomes

$$d\theta=\tan^2\mu d\mu-(1-2\cos^2\mu)^{1/2}\sec^2\mu\,\operatorname{cosec}\mu d\mu \quad . \quad . \quad . \quad . \quad . \quad (20).$$

Integrating (20), we find for the equation to the curve

$$\theta=\sin^{-1}\left(\frac{b}{a}\right)-\sin^{-1}\left(\frac{b}{r}\right)-\frac{b}{c}\pm\frac{b}{(r^2-b^2)^{1/2}}+\frac{1}{c}(2b^2-a^2)^{1/2}\mp\left(\frac{2b^2-r^2}{r^2-b^2}\right)^{1/2}$$
$$-\cot^{-1}\left(\frac{c^2}{2b^2-a^2}\right)^{1/2}+\cot^{-1}\left(\pm\frac{r^2-b^2}{2b_2-r2}\right)^{1/2}\quad . \quad . \quad . \quad . \quad . \quad (21).$$

From (1), (2) and (21) we find

$$z_1=2a\tan^{-1}\left(\frac{c^2}{2b^2-a^2}\right)^{1/2}-\frac{2b^2}{c}+\frac{2b}{c}(2b^2-a^2)^{1/2}\quad . \quad . \quad . \quad (22),$$

the distance the boy walks.

In II and II the curve is convex toward the center of the table, and the motion of the table is not toward the boy for all positions in the curve.

Solved also by *Henry Heaton.*

SOLUTIONS OF "UNSOLVED PROBLEMS,"

Published in No. 1.

14. A cylindrical cask, radius r inches and depth a inches, is full of wine. Through a pipe in the top water can be let in at the rate of b gallons per minute, and through a pipe, radius r inches, in the center of the bottom the mixture can escape at a faster rate when the cask is full. If both pipes be opened at the same instant, how much wine will remain in the cask at the end of t minutes, supposing the two fluids to mingle perfectly?

Solution by DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

Let $V=\pi aR^2$, the volume of the cask; a=its altitude; $q=\frac{b}{60}$=number of gallons of inflow per second; k=the contracted section of the issuing vein; $K=\pi R^2$, the cross-section of the cask; Q=the quantity of liquid, both water and wine, at time t; w=the quantity of wine at that time; x=the variable head or depth of liquid in the cask, and $A=k\sqrt{\left(\frac{2g}{K}\right)}$. Then $x=\frac{Q}{K}$, and the quantity of outflow in an element of time is $k\sqrt{(2gx)}dt, =k\sqrt{(2g)}\sqrt{\left(\frac{Q}{K}\right)}dt, =AQ^{1/2}dt$, and the quantity of inflow being qdt in the same time, we have $-dQ=qdt-AQ^{1/2}dt$ . . (a).

The quantity of wine which flows out in an element of time will be $\frac{w}{Q}$ of the liquid which flows out in that time. Hence $dw=-\frac{wAQ^{1/2}}{Q}dt, =-\frac{Aw}{Q^{1/2}}dt$ . . . . (b).

To integrate (a) let $q-AQ^{1/2}=y$, and we have

$$t=\int_Q^V \frac{-dQ}{q-AQ^{1/2}}=\frac{2(V^{1/2}-Q^{1/2})}{A}+\frac{2q}{A^2}\log\left(\frac{q-AV^{1/2}}{q-AQ^{1/2}}\right) \quad . \ . \ . \quad (c).$$

Substitute dt from (a) in (b) and we have

$$\frac{dw}{w}=\frac{dQ}{\frac{q}{A}Q^{1/2}-Q}.$$ Let $Q^{1/2}=r$, then integrating and observing that for $w=V$, $Q=V=r^2$, we have

$$\frac{w}{V}=\left(\frac{q-AQ^{1/2}}{q-AV^{1/2}}\right)^2; \quad \therefore \quad Q=\frac{[(AV^{1/2}-q)w+qV^2]^2}{A^2V^4},$$

and this value in the equation (c) gives a direct relation between t and w.

To find the limit of Q, make $w=0$, and we find $Q=\frac{q^2}{A^2}$, and this value in equation (c) gives $t=\infty$ as it should.

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20 Three points are taken at random, one in each of the sides of a given triangle; find the chance that the triangle formed by joining them is acute.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Let ABC be the given triangle, M, N, P the random points Draw AD perpendicular to BC, MS and MT perpendicular to AM and BC, and MR to MN. When the random point in BC falls on BD, as at M′ draw M′S′, M′T′ and M′R′ perpendicular to AM′, BC and M′P. If N falls on CS, the angle M of the triangle MNP will be obtuse if P falls anywhere on AB; and if N falls on ST the angle M will be obtuse if P falls anywhere on BR.

Put CM=x, CN=y, BR=z, CS=u, CT=v, BM′=x', BP=y', CR′=z', BS′=u', BT′=v', CD=m, BD=n, cotAMD=t, cotAM′D=t', cotC=t_1, cotB=t_2. Then we have $u=\frac{tx}{t\cot C+\sin C}$, $v=x\sec C$, $x=b(\cos C-t\sin C)$, $u'=\frac{t'x'}{t'\cos B+\sin B}$, $v'=x'\sec B$, $x'=c(\cos B-t'\sin B)$, $dx=-b\sin C\,dt$, $dx'=-c\sin B\,dt'$, $y\sin C : x-y\cos C :: a-x-z\cos B : z\sin B$, whence

$$z=\frac{(a-x)x\sin B\sin C}{\cos A(x\cos B+y\cos A)}-\frac{(a-x)\cos C}{\cos A}.$$

Similarly we find $z' = \frac{(a-x')x' \sin B \sin C}{\cos A(x' \cos C + y' \cos A)} - \frac{(a-x')\cos B}{\cos A}$.

Hence the chance that the angle M is obtuse is

$$M_0 = \frac{1}{abc}\int_0^m \left(\int_0^u c dy + \int_u^v z dy\right) dx + \frac{1}{abc}\int_0^n \left(\int_0^{u'} b dy' + \int_{u'}^{v'} z' dy'\right) dx',$$

$$= \frac{b \sin C}{a}\int_0^{t_1}\left[\frac{t \cos B \sin C}{\cos A} - \frac{\cos B \cos C}{\cos A} + \frac{\sin^2 B \sin^2 C}{\cos^2 A}(t+t_2)(t_1-t)\log\left(\frac{t+\tan C}{t+t_2}\right)\right]dt$$

$$+\frac{b \sin C}{a}\int_0^{t_2}\left[\frac{t' \sin B \cos C}{\cos A} - \frac{\cos B \cos C}{\cos A} + \frac{\sin^2 B \sin^2 C}{\cos^2 A}(t'+t_1)(t_2-t')\log\left(\frac{t'+\tan B}{t'+t_1}\right)\right]dt',$$

$$= \frac{4}{3} + \frac{\cos B \cos C}{\cos A} - \frac{1}{2\cos A} + \frac{\cos A}{3\cos B \cos C} + \tfrac{1}{3}(3+2\cot A \tan B)\tan^2 B \log \sin B$$

$$+\tfrac{1}{3}(3+2\cot A \tan C)\tan^2 C \log \sin C - \tfrac{4}{3}\tan^2 A \log \sin A.$$

The expressions for the chances, N_0 and P_0, that the angles N and P will be obtuse, are similar to that for M_0. Hence, since a triangle can have but one obtuse angle, the chance that the triangle is obtuse is $M_0 + N_0 + P_0$, and the chance that it is acute is

$$p = 1 - (M_0 + N_0 + P_0),$$

$$= \frac{1}{3}\left(5 + \frac{2\sin A \tan A}{\sin B \sin C}\right)\tan^2 A \log \operatorname{cosec} A + \frac{1}{3}\left(5 + \frac{2\sin B \tan B}{\sin A \sin C}\right)\tan^2 B \log \operatorname{cosec} B$$

$$+\frac{1}{3}\left(5 + \frac{2\sin C \tan C}{\sin A \sin B}\right)\tan^2 C \log \operatorname{cosec} C - \left(\frac{\cos A \cos B}{\cos C} + \frac{\cos A \cos C}{\cos B} + \frac{\cos B \cos C}{\cos A}\right)$$

$$+\frac{1}{2}\left(\frac{1}{\cos A} + \frac{1}{\cos B} + \frac{1}{\cos C}\right) - \frac{1}{3}\left(\frac{\cos C}{\cos A \cos B} + \frac{\cos B}{\cos A \cos C} + \frac{\cos A}{\cos B \cos C}\right) - 3.$$

Cor.—When $a=b=c$, $p = \frac{1}{2}\left(27 \log \frac{4}{3} - 7\right)$.

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28. Find the average distance between two points taken at random in the surface of a given rectangle, one on each side of a diagonal.

Solution by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Let M be the mean distance required, M_1 the mean distance between two points taken indiscriminately in the rectangle and M_2 the mean distance between two points taken both on the same side of a diagonal.

Then (*Williamson's Integral Calculus*, p. 305,)

$$M = 2M_1 - M_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

But (*Williamson's Integral Calculus*, p. 348,)

$$M_1 = \frac{1}{15}\left[\frac{a^3}{b^2} + \frac{b^3}{a^2} + d\left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2}\right) + \frac{5}{2}\left(\frac{b^2}{a}\log\frac{a+d}{b} + \frac{a^2}{b}\log\frac{b+d}{a}\right)\right];$$

and (*Educational Times Reprint*, vol. xxiii, p. 92,)

$$M_2 = \frac{1}{15}\left[\left(\frac{a^2}{d} + \frac{d^2}{a}\right)(\cos B + \sin^2 B \log\cot\tfrac{1}{2}B) + \left(\frac{b^2}{d} + \frac{d^2}{b}\right)(\cos A + \sin^2 A \log\cot\tfrac{1}{2}A)\right],$$

$$= \frac{1}{15}\left[\left(\frac{a^2}{d} + \frac{d^2}{a}\right)\left(\frac{a}{d} + \frac{b^2}{d^2}\log\frac{a+d}{b}\right) + \left(\frac{b^2}{d} + \frac{d^2}{b}\right)\left(\frac{b}{d} + \frac{a^2}{d^2}\log\frac{b+d}{a}\right)\right].$$

Substituting in (1),

$$M=\frac{1}{15}\Big[2d\left\{2-\frac{a^2}{b^2}-\frac{b^2}{a^2}\right\}-\frac{a^3+b^3}{a^2+b^2}+\frac{2a^3}{b^2}+\frac{2b^3}{a^2}+\left\{\frac{4b^2}{a}-\frac{a^2b^2}{d^3}\right\}\log\left\{\frac{a+d}{b}\right\}$$
$$+\left\{\frac{4a^2}{b}-\frac{a_2b^2}{d^4}\right\}\log\left\{\frac{b+d}{a}\right\}\Big].$$

Cor.—If $a=b$, $M=\frac{1}{30}a[6+(16-\sqrt{2})\log(1+\sqrt{2})]$.

This problem was solved in a very elegant manner by *E. B. Seitz.*

LIST OF CONTRIBUTORS.

Solutions of the Problems published in No. 1 have been received as follows:

From E. B. Seitz, Greenville, Ohio, 8, 12, 18, 19, 20, 22, 24, 26, 28, 29, 30, 31, 32, 33 and 34, and 14, 20 and 28 of "Unsolved Problems;" B. F. Burleson, Oneida Castle, New York, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 and 14; James McLaughlin, Mantorville, Minnesota, 2, 4, 6, 7, 8, 9, 10, 11, 12 and 13; William Hoover, Bellefontaine, Ohio, 1, 2, 3, 4, 8, 9, 10, 14 and 20; K. S. Putnam, Rome, New York, 1, 2, 3, 4, 5, 6, 8 and 15; Dr. S. F. Bachelder South Boston, Massachusetts, 1, 2, 4, 8, 10, 11, 12 and 13; Henry Heaton, B. S., Sabula, Iowa, 15, 24, 25, 26, 30 and 34; F. P. Matz, B. E., B. S., Reading, Pennsylvania, 2, 3, 10, 14, 20 and 25; Marcus Baker, U. S. Coast Survey Office, Washington, D. C., 1, 3, 8, 9, 10 and 14; Miss Christine Ladd, B. A., Union Springs, New York, 4, 7, 9 and 17; O. D. Oathout, Read, Iowa, 1, 2, 3 and 10; W. L. Harvey, Maxfield, Maine, 1, 2, 5 and 6; David Wickersham, Wilmington, Ohio, 7, 9, 11 and 12; Prof. David Trowbridge, M. A., Waterburg, New York, 7, 16 and 25; G. M. Day, Lockport, New York, 16, 17 and 20; D. J. McAdam, Washington, Pennsylvania, 10, 19 and 20; Walter Siverly, Oil City, Pennsylvania, 22 and 25; Dr. S. H. Wright, M. A., Ph. D., Penn Yan, New York, 12 and 19; Prof. DeVolson Wood, Hoboken, New Jersey, 25, and 14 of "Unsolved Problems;" Mrs. Anna T. Snyder, Orange, Indiana, 2 and 6; Theo. L. DeLand, U. S. Treasury Department, Washington, D. C., 5 and 6; S. C. Gould, Manchester, New Hampshire, 3; Prof. W. W. Johnson, Annapolis, Maryland, 24; Dr. David S. Hart, Stonington, Connecticut, 21; D. W. K. Martin, Webster, Ohio, 2; E. P. Norton, Allen, Michigan, 13; I. H. Turrell, Cumminsville, Ohio, 18; H. T. J. Ludwick, Mt. Pleasant, North Carolina, 16; George Eastwood, Saxonville, Massachusetts, 19; Chas. H. Kummell, Detroit, Michigan, 30.

JUNIOR PROBLEMS.

35.—Proposed by Orlando D. Oathout, Read, Clayton County, Iowa.

A and B propose to trade horses. A asks $20 "to boot," and B asks $10. They finally agree to "split the difference" How much "boot money" should A receive?

36.—Proposed by J. B. Sanders, Bloomington, Monroe County, Indiana.

A merchant bought 400 yards of cloth at $4 per yard, payable in 3 months, and after holding it for 15 days sold it at $4.25 per yard, receiving therefor a note payable in 4 months. When the purchase money became due, he had this note discounted at the bank to meet it. What did he gain by the transaction? What would he have gained if he had borrowed at 6 per cent. interest, until the maturity of the note he had received, sufficient to pay for the cloth, and why should there be any difference in the results?

37.—Proposed by B. F. Burleson, Oneida Castle, Oneida County, New York.

A parsimonious farmer has a board $6\frac{1}{2}$ feet long and 2 feet wide which he wishes to cut and form into a square without any waste of lumber. How must he cut the board and arrange the pieces?

38.—Proposed by Prof. Edward Brooks, M. A., Ph. D., Principal Pennsylvania State Normal School, Millersville, Lancaster County, Pennsylvania.

A man gave me his note for $500 payable in 8 years, interest at 6 per cent.; if he pays the interest annually, to what rate is this equivalent if the same amount of interest had been paid at the end of the time?

39.—Proposed by W. B. BATES, Earlville, LaSalle County, Illinois.

If Dr. A kills 3 patients out of 7; Dr. B, 4 out of 13; and Dr. C, 5 out of 19, what chance has a sick man for his life who employs all three of these doctors at the same time?

40.—Proposed by J. G. WALTON, Covington, Kenton County, Kentucky.

Two circles have the same center. If a chord, length $2a$, be drawn across the larger circle, it will just touch the circumference of the smaller. Required the area of the ring.

41.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find x and y from the equations

$$xy+(x^2+y^2)(1+xy+x^2y^2+x^3y+xy^3)=87,$$
$$xy(x^2+y^2)(x^2+xy+y^2)(x^2+y^2+xy+xy^3+x^3y)=1190.$$

42.—Proposed by BENJAMIN HEADLEY, Dillsborough, Dearborn County, Indiana.

Cut a board 3 feet wide and 7 feet long into three pieces, so that they will make a square.

43.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Two men, A and B, plow a rectangular field a rods long and b rods wide, A going ahead all the time. What part of the field does each plow, estimating n furrows to the rod?

44.—Proposed by WINFIELD V. JEFFRIES, Instructor in Mathematics, Vermillion Institute, Hayesville, Ashland County, Ohio.

From a given point without a circle, to draw a secant that shall be bisected by the circumference.

45.—Proposed by Mrs. ANNA T. SNYDER, Orange, Fayette County, Indiana.

A father having 640 acres of land in a circle gives to each of his six sons a circular farm touching the circumference of the circle and also the circumferences of the farms of two of his brothers, the six farms being equal in size. Required the number of acres in the farm of each of the sons, the number remaining to the father, the number remaining in the center and the number in each of the other six equal pieces.

46.—Proposed by Prof. J. F. W. SCHEFFER, College of St. James, Washington County, Maryland.

The sides of a plane triangle are proportional to the roots of the cubic equation

$$x^3-ax^2+bx-c=0.$$

Find the sum of the cosines of the angles of the triangle.

47.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Monroe County, Indiana.

The number of diagonals that can be drawn in a certain polygon is equal to 7 times the number of sides. How many sides has the polygon?

48.—Proposed by JOSEPH FICKLIN, M. A., Ph. D., Professor of Mathematics and Astronomy, University of the State of Missouri, Columbia, Boone County, Missouri.

Find the radius of a sphere that shall circumscribe four equal spheres which touch each other.

49.—Proposed by Dr. S. F. BACHELDER, South Boston, Massachusetts.

P and Q owned a triangular piece of land, the sides of which were 20, 18 and 4 chains. The longest side bordered on a road which ran east and west. They agreed to divide it by a line parallel to the shortest side so that Q, who took the resulting trapezoid, should have $\frac{4}{5}$ of the land. A surveyor located the line; then P, setting a stake at the middle of it, proposed that a new line should be run north and south through that point, so as to give both a square corner on the road. Q assented; did he grain or lose, in area, by the change of line, and how much?

50.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In any plane triangle ABC the angles B and C are bisected, the bisectors meeting the sides in B′ and C′ respectively. Join B′ and C′ and from any point P in B′C′ let perpendiculars, p_a, p_b and p_c, fall upon the sides a, b and c respectively; prove that

$$p_a = p_b + p_c.$$

51.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

There is a series of right-angled triangles whose legs differ by unity only. Calling the one whose sides are 3, 4, 5 the *first* triangle, it is required to find general expressions for the sides of the *n*th triangle, and compute the sides of the 100th triangle.

52.—Proposed by JAMES MCLAUGHLIN, Mantorville, Dodge County, Minnesota.

Three circles, radii a, b, c, are drawn in a triangle, each circle touching the other two and two sides of the triangle. Find the sides of the triangle.

53.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

The towns C and D are 40 miles apart. A set out from C to travel to D at the same time that B started from D to go to C. A *overtook* a drove of sheep moving forward at the rate of 2 miles an hour just 10 minutes *after* he crossed a creek known to be 10 miles from C; B arrived at C 3 hours and 55 minutes after he *met* the same drove. B was *overtaken* by an express proceeding at the rate of 5 miles an hour just 5 minutes *before* he came to an inn 12 miles from D; A arrived at D 8 hours and 20 minutes after he *met* the express. Required the hourly speed of A and B.

54.—Proposed by E. J. ROWAN, Shawnee, Perry County, Ohio.

What length of line, fastened to a point in the circumference of a circle whose area is one acre, will allow an animal to graze upon just one acre outside of the circle?

55.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, Erie, Erie County, Pennsylvania.

Given $y = x + ax^2 + bx^3 + cx^4 + dx^5 + ex^6 + fx^7 + gx^8 + hx^9 + ix^{10} + jx^{11} + kx^{12} + lx^{13} + mx^{14} + nx^{15} + px^{16} + \&c. \quad . \quad . \quad (1),$

and $x = y + Ay^2 + By^3 + Cy^4 + Dy^5 + Ey^6 + Fy^7 + Gy^8 + Hy^9 + Iy^{10} + Jy^{11} + Ky^{12} + Ly^{13} + My^{14} + Ny^{15} + Py^{16} + \&c. \quad . \quad . \quad (2);$

to find $A, B, C, D, E, F, G, H, I, J, K, L, M, N, P$ in terms of $a, b, c, d, e, f, g, h, i, j, k, l, m, n, p$.

Solutions of these problems should be received by September 1, 1878. Ten copies of the VISITOR will be given for the best, complete, correct solution of the prize problem, and eight for the second best solution.

SENIOR PROBLEMS.

56—Proposed by GEORGE EASTWOOD, Saxonville, Middlesex County, Massachusetts.

In a plane, the equation of a straight line in terms of the perpendicular (p) from the origin and the angle (θ) which it makes with the axis being $y\sin\theta + x\cos\theta = p$; prove that the same form holds for the equation of a great circle of the sphere, when x, y and p are put for $\tan x$, $\tan y$ and $\tan p$.

57.—Proposed by OSCAR H. MERRILL, Mannsville, Jefferson County, New York.

Prove that the cube of any given number is greater than the product of any other three numbers whose sum is three times the given number.

58.—Proposed by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

An ellipse and a parabola have a common focus and the other focus of the ellipse moves on the directrix of the parabola. Show that the points of contact of a common tangent subtends a right angle at the common focus.

59.—Proposed by Dr. S. H. WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, Yates County, New York.

When the sun's declination was δ north he rose β° farther south than when his declination was δ' north. Required my latitude.

60.—Proposed by GEORGE LILLEY, Kewanee, Henry County, Illinois.

Find the minimum eccentricity of an ellipse capable of resting in equilibrium on a perfectly rough inclined plane, inclination β.

61.—Proposed by Miss CHRISTINE LADD, B. A., Professor of Natural Sciences and Chemistry, Howland School, Union Springs, Cayuga County, New York.

If ABC be a triangle inscribed in a conic, $P'P''P'''$ the points in which its sides meet the directrix of the conic, and $Q'Q''Q'''$ the poles of focal chords through $P'P''P'''$, then will AQ', BQ'', CQ''' meet in a point.

62.—Proposed by F. P. MATZ, B. E., B. S., Mathematical Editor *National Educator*, Reading, Berks County, Pennsylvania.

In an equilateral triangle ABC lines are drawn as in Problem 8; find the average area of the equilateral triangle formed by the intersections of the lines.

63.—Proposed by ISAAC H. TURRELL, Cumminsville, Hamilton County, Ohio.

Within the space enclosed by three given circles which touch each other externally it is required to inscribe, geometrically, three circles each of which shall touch the other two and also two of the given circles. Analytical solutions also desired.

64.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

If there be two equations in x, (which for greater simplicity may be supposed to be of the same degree n) find the most general form of M=a rational integral function of the coefficients of these equations such that Mx, Mx^2, ... Mx^{n-1} shall each of them also be rational functions of the same.

65.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

Find six square numbers whose sum is a square, and the sum of their roots plus the root of their sum a square.

66.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Middlesex County, Massachusetts.

What are the probabilities at a game of a given number of points, but at which there is only one person who is the actual player? When the player is successful he counts a point, but when he is unsuccessful he loses all the points which he has made and adds one point to the score of his opponent.

67.—Proposed by Dr. JOEL E. HENDRICKS, M. A., Editor of the *Analyst*, Des Moines, Polk County, Iowa.

Suppose the earth to be an indefinitely thin spherical shell, equatorial radius, rotary velocity and gravitating force the same as at present except that gravity is assumed to be all concentrated at the center, and none in the shell; and suppose a particle at the equator, and on the inside of the shell, to be separated from the shell, and to move henceforth from its centrifugal force, resulting from its motion while attached to the shell, and from the gravitating force at the earth's center. Required the axes of the ellipse the particle will describe, and the time required for its return to the same point in space at which it was detached from the shell.

68.—Proposed by Prof. DAVID TROWBRIDGE, M. A., Waterburg, Tompkins County, New York.

A sphere is divided at random by a plane, and then two points are taken at random within the sphere; find the chance that both points are on the same side of the plane.

69.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Monroe County, Indiana.

Given the radius of Mars, 2250 miles, and the radius of the orbit of its inner satellite, 5800 miles; to determine whether the latter can have an elastic atmosphere, supposing its diameter to be 45 miles, and its density equal to that of the primary.

70.—Proposed by JOHN H. ADAMS, Cochranton, Crawford County, Pennsylvania.

In digging a well 6 feet in diameter a log 3 feet in diameter was found lying directly across the center of the well. How many cubic feet of the log must be removed from the well?

71.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Middlesex County, Massachusetts.

Given the skill of two billiard players at the three-ball game, to find the chance of the better player gaining the victory if he gives the other a *grand discount.*

—[From *Our Schoolday Visitor*, vol. xv, p. 220.

72.—Proposed by D. J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania.

The center of an epicycle whose radius is $\frac{1}{12}a$ revolves on a deferent, radius a, with uniform angular velocity v; a particle revolves in the epicycle so that the radius drawn in the small circle to the particle is always parallel to itself. Supposing the particle to be moving under the influence of a force at the center of the large circle, find the law of the force.

73.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find the average distance between two points taken at random in the surface of a given semicircle.

74.—Proposed by Rev. W. J. WRIGHT, M. A., Ph. D., Member of the London Mathematical Society, Carlisle, Cumberland County, Pennsylvania.

Transform the Eulerian integral $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ to the Legendreian, $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$

75.—Proposed by Prof. H. A. WOOD, M. A., Principal Coxsackie Academy, Coxsackie, Greene Co., N. Y.

A sphere 4 inches in diameter, specific gravity 0.2, is placed 10 feet under water. If left free to move, what will be its velocity at the surface of the water, and what will be the maximum hight it will attain?

76.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

Prove that the equation to the sums and the equation to the products of the roots of an equation of the nth degree taken two and two together may each be put under the form of a determinant of the order $\frac{1}{2}(n-1)$ or of the order $\frac{1}{2}n$ according as n is odd or even, and write down the determinant of the third order which equated to zero is the equation (of the degree 21) to the binary products of an equation of the 7th degree clear of any irrelevant factor.

77.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Two points are taken at random in the surface of the quadrant of a circle, and a line drawn through them. Find the chance (1) that the line intersects the arc in two points, (2) that it intersects the arc in one point, and (3) that it does not intersect the arc.

78.—Proposed by DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

One end of a fine, inextensible string is attached to a fixed point, and the other end to a point in the surface of a homogenous sphere, and the ends brought together, the center of the sphere being in a horizontal through the ends of the string, and the slack string hanging vertically. The sphere is let fall and an angular velocity imparted to it at the same instant, the sphere winding up the string on the circumference of a great circle until it winds up all the slack when it suddenly begins to ascend, winding up the string, the sphere returning just to the starting point. Required the initial angular velocity, the tension of the string during the ascent of the sphere, the initial upward velocity of the center of the sphere, and the time of the movement.

79.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find the average distance of the center of an ellipse, axes $2a$ and $2b$, from its circumference.

80.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

A chord is drawn through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that it will intersect the first chord.

81.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

A fixed hemispherical bowl, radius R, is full of water and contains a heavy cylindric rod, length $2a$, radius r and specific gravity ρ, having one end against its concave surface, and resting on its rim. Determine the inclination of the rod.

82.—Proposed by JOHN M. WILT, M. A., Fort Wayne, Allen County, Indiana.

An unknown cone is cut at random by a plane; find the chance that the section is an ellipse.—From the *Normal Monthly*, vol. iii, p. 108.

83.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find the average distance of one corner of a rectangular solid, edges a, b, c, from all points within it.

84.—Proposed by E. P. NORTON, Allen, Hillsdale County, Michigan.

A fox starts from a point 100 rods due north from a hound, and runs due west at the rate of 10 miles an hour; at the same instant the hound starts in pursuit, at the rate of 15 miles an hour, always keeping in a direct line between his starting point and the fox. Required the equation of the *curve* described by the hound, and the distance he runs to catch the fox.—From the *Normal Monthly*, vol. iii, p. 85.

85.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

Let X be the coefficient of any power of a in the expansion of

$$\frac{1}{(1-xa)(1-x^2a)(1-x^3a)\ldots\ldots(1-x^ia)}$$

where i is any integer, and let X be arranged according to the powers of x; prove that its coefficient will form a series reading the same from left to right as from right to left, possessing this property that as we pass from either end toward the central term or terms the coefficients may increase or remain unaltered but can never decrease.

86.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Find the average distance between two points taken at random within a rectangular solid, edges a, b, c.

87.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Erie, Erie Couty, Pennsylvania.

A cask containing a gallons of wine is placed upon another containing b gallons of brandy. Water runs in at the top of the wine cask at the rate of m gallons per minute, the mixture escapes into the brandy cask at the same rate, the mixture in the brandy cask escapes at a like rate into a tub containing c gallons of water and the tub overflows. Find the quantity of each fluid in the tub at the end of t minutes supposing them to mingle perfectly.

Solutions of these problems should be received by September 1, 1878.

UNSOLVED PROBLEMS,

By ARTEMAS MARTIN, M. A., Erie, Pa.

31. Solve the equations

$$c\sqrt{(b+c+x)(a+c+y)}-c\sqrt{(a+b+c)(c+x+y)}=2\sqrt{abxy},$$
$$b\sqrt{(b+c+x)(a+b+z)}-b\sqrt{(a+b+c)(b+x+z)}=2\sqrt{acxz},$$
$$a\sqrt{(a+c+y)(a+b+z)}-a\sqrt{(a+b+c)(a+y+z)}=2\sqrt{bcyz},$$

by quadratics.

32. Two rods, lengths $2a$ and $2b$, have their middle points connected by a string, length c. If they be thrown on a level floor, what is the chance of the rods being crossed?

33. One end of a fine string, length l, is attached to a wooden sphere, radius r and specific gravity ρ, and the other end is fastened to a point in the bottom of a river, depth a and velocity v. Determine the position of the sphere.

34. A cubic equation is written down at random; what is the chance that its roots are the sides of a real triangle?

35. A cask in the form of a frustum of a cone, radius of top r inches, radius of bottom R inches and depth a inches is full of water. Required the time of emptying through a pipe in the bottom, radius m inches, the quantity of water in the cask at the end of any time t, and the velocity of discharge. A rigorous solution from first principles is desired.

36. What is the chance that the roots of a biquadratic equation written down at random are all real?

37. The first of two casks contained a gallons of wine and the second contained b gallons of water. Part of the water was poured into the first cask, then part of the mixture was poured into the second, and part of the mixture in the second poured back into the first. Required the probability that not more than $\frac{1}{n}$ of the contents of the first cask is wine.

38. A sportsman saw a duck in a circular pond, which dodged behind a tree growing in it. The sportsman ran along the edge of the pond, but the duck kept behind the tree all the time and swam towards the shore. Required the equation of the *curve* the duck described, and the distance it swam to reach the shore.

39. Two men start from the same side of a square field and walk across it in random directions. Find (1) the chance that both men will cross the opposite side of the field and (2) the chance that their paths will intersect within the field.

40. The first of two casks contains a gallons of wine, and the second b gallons of water. From the first is poured into the second as many gallons as it already contains, and then as much is poured from the second into the first as was left in the first. How much wine remains in the second cask after n such operations?

41. Three circles, radii a, b, c, touch each other externally. A point is taken at random in each circle. Find the chance that the triangle formed by joining the points is acute.

42. Suppose a hawk a yards north, and an eagle b yards south, of a pigeon. The pigeon flies due east at the rate of m miles an hour; the hawk flies continually towards the pigeon; the eagle flies continually towards the hawk, and the eagle catches the hawk at the same instant that the hawk catches the pigeon. Required the equation to the *curve* the eagle describes, and the distance each has flown when the hawk and pigeon are caught.

43. The first of two casks contained a gallons of wine and the second contained b gallons of water. Part of the water was poured out of the second cask and then it was filled up out of the first cask, and the deficiency in the first supplied with water. After n such operations, what is the probability that less than $\frac{1}{m}$ of the contents of the second cask is wine?

44. The velocity of a river a yards wide is v miles per hour. A deer that can swim n miles an hour in still water starts to swim across the river, all the time aiming for a point in the bank directly opposite the point he started from. At the same instant a dog, b yards up stream from the deer, that can swim m miles an hour in still water, starts in pursuit, and swims continually towards the deer. Required the equation to the *curve* described by the deer; the equation to the *curve* described by the dog, and the distance each swims before the deer is caught.

45. A heavy flexible cable, length a, slides endwise down a smooth inclined plane, inclination β, the lower end of the plane being at a distance above the ground greater than the length of the cable. Required the velocity of the cable at any time t during the motion, and the form of the cable at the moment the upper end leaves the lower end of the plane.

EDITORIAL NOTES.

We are grateful for the very liberal patronage extended to the first No. of the VISITOR, and hope that the present No. will meet with a like favorable reception at the hands of the mathematical public. It will be our constant aim to make each No. more desirable than the one preceding it.

Mr. E. B. SEITZ and Prof. BENJAMIN PEIRCE have our sincere thanks for valuable assistance.

No. 3 will be published about the 1st of January, 1879; it will contain about 32 pages, and the price will be 50 cents. Persons desiring to secure copies should send their orders at an early date, as only a limited number will be printed.

Copies of No. 1 cannot be supplied, as the whole edition is exhausted.

The VISITOR will be published semi annually, and perhaps quarterly, as soon as our subscription list is large enough to meet the expense.

We have left a few copies of *Our Schoolday Visitor Mathematical Annual* for 1871. Price, 25 cents.

The first prize is awarded to E. B. SEITZ, Greenville, O., and the second to HENRY HEATON, Sabula, Iowa.

Send all orders to ARTEMAS MARTIN, Lock Box 11, Erie, Pa.

NOTICES OF BOOKS AND PERIODICALS.

Lectures on the Elements of Applied Mechanics. By Morgan W. Crofton, F. R. S., Professor of Mathematics and Mechanics at the Royal Military Academy. 8 vo. pp. 107. London, England: C. F. Hodgson & Son.

"A synopsis of a course of Lectures on the elements of the Theory of Structures and the Strength of Materials, forming the first part of the Course of Applied Mechanics, at present studied by the Gentleman Cadets of the Royal Military Academy." The subject is admirably treated in an elementary manner, without the Calculus, and illustrated by numerous examples.

The Principles of Elementary Mechanics. By DeVolson Wood, Professor of Mathematics and Mechanics in the Stevens Institute of Technology. 12mo., pp. 351, Price $2.00. New York: John Wiley & Sons.

A hurried examination of this work convinces us that it is one of the best treatises on Elementary Mechanics we have yet seen. It is written in Prof. Wood's lucid style, and the different subjects are amply elucidated by a great number of interesting and well-selected problems and solutions.

Tracts Relating to the Modern Higher Mathematics. Tract No. 2. Trilinear Co-ordinates. By Rev. W. J. Wright, Ph. D., Member of the London Mathematical Society. 8vo., paper, pp. 77. London, England: C. F. Hodgson & Son.

A full and clear presentation of the subject of Trilinear Co-ordinates, with application to examples.

A list of writings relating to the method of Least Squares, with Historical and Critical Notes. By Mansfield Merriman, Ph. D., Instructor in the Sheffield Scientific School of Yale College. From the Transactions of the Connecticut Academy. Vol. IV, 1877. 8vo., pamphlet, pp. 82.

This is a work of great value to students. It contains the titles of 408 papers, books and parts of books, written in eight languages, ranging in date from 1722 to 1876.

On the Transcendental Curves whose equation is $\sin y \sin my = a \sin x \sin nx + b$. By H. A. Newton, LL. D., Professor of Mathematics in Yale College, and A. W. Phillips, Ph. D., Tutor in Matehmatics in Yale College. From the Transactions of the Connecticut Academy. 8vo., pamphlet, 11 pp. of text and 24 plates.

An interesting discussion of some of the curves represented by the equation given in the title.

On a New Method of Obtaining the Differentials of Functions with especial reference to the Newtonian Conception of Rates or Velocities. By J. Minot Rice, Professor of Mathematics in the United States Navy and W. Woolsey Johnson, Professor of Mathematics in Saint John's College, Annapolis, Maryland. Revised Edition. 16mo., pamphlet, pp. 32. Price 50 cents. New York: D. Van Nostrand.

On the Part of the Motion of the Lunar Perigee which is a Function of the Mean Motions of the Sun and Moon. By G. W. Hill, Ph. D., Assistant in the Office of the American Ephemeris and Nautical Almanac. Quarto, pp. 28.

An able investigation of an important problem in Astronomy.

The Method of Least Squares applied to a Hydraulic Problem. By Mansfield Merriman, Ph. D., Instructor in the Sheffield Scientific School. 8vo., pamphlet, pp. 9. Reprinted from the Journal of the Franklin Institute for October, 1877.

The problem considered is that of determining the velocity of a river at different depths below the surface of the water.

The History of Malfatti's Problem. By Marcus Baker, U. S. Coast Survey. 8vo., pamphlet, pp. 10. From the Bulletin of the Philosophical Society of Washington.

In this interesting paper Mr. Baker has given an account of several solutions of this famous problem, and a list of the authors who have considered it.

The Educational Times, and Journal of the College of Preceptors. London, England: C. F. Hodgson & son. The mathematical department continues under the able editorship of W. J. C. Miller, B. A. Each number contains three or four double-column pages of problems and solutions. Many of the leading mathematicians of this country and Europe are numbered among its contributors.

Reprint of the Mathematics from the Educational Times. Same publishers. Issued in half-yearly volumes of 112 pp., 8vo., boards. Contains besides the mathematics published in the *Times* about as much more original matter. Vol. xxvi contains solutions of 86 problems and 2 papers; vol. xxvii contains solutions of 65 problems and 9 papers. The *Reprints* are rich in "Probability" and "Average" solutions.

The editor of the Visitor can furnish the *Times* at $2 a year, and the *Reprint* at $1.85 per vol.

The American Journal of Pure and Applied Mathematics. Prof. J. J. Sylvester, LL. D., F. R. S., Editor in Chief; W. E. Story, Ph. D., (Leipsic) Associate Editor; with the co-operation of Profs. Benj. Peirce, LL. D., F. R. S., of Harvard University, Simon Newcomb, LL. D., Ph. D., of the U. S. Naval Observatory, and H. A. Rowland, C. E. Published under the auspices of the Johns Hopkins University, Baltimore, Maryland. To be issued in quarterly numbers of 96 quarto pages. Price $5 a year. The first number will appear in January, 1878.

The illustrious names connected with the *Journal* are a sufficient guaranty that the matter will be of the highest order of excellence.

The Analyst. A Journal of Pure and Applied Mathematics. DesMoines, Iowa: Edited and published by J. E. Hendricks, M. A. Bi-monthly; each No. contains 32 pp. Price $2 a year. No. 1, vol. 5, contains several valuable papers, and the usual number of problems and solutions.

The *Analyst* is conducted with marked ability, and should have a wide circulation.

The Wittenberger, published at Sprinfield, Ohio, contains a mathematical department ably conducted by William Hoover.

The *Wittenberger* is a neat monthly magazine, devoted to the interest of Wittenberg College. Price $1.10 per year.

Educational Notes and Queries, published monthly, except in the vacation months of July and August, contains a mathematical department devoted to problems, solutions and mathematical notes. Prof. W. D. Henkle, Salem, Ohio, Editor and Publisher. Price $1.

The National Educator, published monthly at Kutztown, Pennsylvania, by A. B. Urick, contains a department of Science and Practical Mathematics, conducted by F. P. Matz, B. E., B. S.

The Yates County Chronicle, published weekly at Penn Yan, New York, by the Chronicle Publishing Co., contains a Mathematical Department conducted by Dr. S. H. Wright, M. A., Ph. D.

The Dr. is publishing a valuable collection of formulas in Trigonometry, Astronomy, &c. $2 a year.

The Jefferson County Journal, published at Adams, New York, by Hatch & Allen, contains a Mathematical department conducted by O. H. Merrill.

The Maine Farmers' Almanac for 1878, Masters & Livermore, Hallowell, Maine, contains the usual Puzzle and Mathematical Departments. Solutions of the 7 questions proposed last year are given and 5 new ones proposed.

CORRIGENDA.

Page 18, solution of problem 8, line 1, for "CE" read CD.

Page 23, solution of problem 16, line 5, for "$\frac{4(-1)_m}{m^2}$" read $\frac{4(-1)^m}{m^2}$.

Page 24, solution of problem 18, line 4 from the end of the solution, for "y" read p.

Page 27, solution of problem 23, line 12 from the end of the solution, for "p" read p_m.

Page 39, second line from the end of the prize solution, for "II and II" read II and III.

Page 40, line 4 from the bottom, in the value of u, for "cotC" read cosC.

Vol. I. JANUARY, 1879. No. 3.

THE MATHEMATICAL VISITOR.

EDITED AND PUBLISHED BY

ARTEMAS MARTIN, M. A.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

PRICE, FIFTY CENTS.

ERIE, PA.:
THE MORNING DISPATCH STEAM BOOK AND JOB PRINTING HOUSE.
1879.

THE MATHEMATICAL VISITOR.

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Vol. 1. **JANUARY, 1879.** **No. 3.**

JUNIOR DEPARTMENT.

SOLUTIONS OF PROBLEMS PROPOSED IN No. 2.

35.—Proposed by ORLANDO D. OATHOUT, Read, Clayton County, Iowa.

A and B propose to trade horses. A asks $20 "to boot," and B asks $10. They finally agree to "split the difference." How much "boot money" should A receive?

I.—Solution by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

B's horse plus $20 is A's valuation of his own horse, and B's horse minus $10 is B's valuation of A's horse; hence if they "split the difference," the value of A's horse is, by the agreement, one-half the sum of A's and B's valuations, or B's horse plus $5; therefore B should pay A $5.

II.—Solution by OSCAR H. MERRILL, South Rutland, Jefferson Co., N. Y.; K. S. PUTNAM, Rome, Oneida Co., N. Y.; THEO. L. DELAND, U. S. Treasury Department, Washington, D. C.; WILLIAM WILEY, Detroit, Wayne Co., Mich.; and D. W. K. MARTIN, Webster, Darke Co., Ohio.

Since they are $30 apart at first, each will have to throw off $15; hence A should receive $5.

Answered in a similar manner by the *Proposer, B. F. Burleson, J. R. Fagan, V. Webster Heath, E. P. Norton, John I. Clark,* Prof. *F. P. Matz* and *Walter S. Nichols.*

III.—Solution by GAVIN SHAW, Kemble, Ontario, Canada.

The difference between A and B is $\$20+\$10=\$30$. If they split the difference, A must come down $15; consequently the "boot" that A should receive is $5.

36.—Proposed by J. B. SANDERS, Bloomington, Monroe County, Indiana.

A merchant bought 400 yards of cloth at $4 per yard, payable in 3 months, and after holding it for 15 days sold it at $4.25 per yard, receiving therefor a note payable in 4 months. When the purchase money became due, he had this note discounted at the bank to meet it. What did he gain by the transaction? What would he have gained if he had borrowed at 6 per cent. interest, until the maturity of the note he had received, sufficient to pay for the cloth, and why should there be any difference in the results?

Solution by K. S. PUTNAM, Rome, Oneida County, N. Y.; Prof. FRANK ALBERT, Millersville, Lancaster Co., Pa.; and E. B. SEITZ, Greenville, Darke Co., Ohio.

His gain was $100, less the interest on $1700 for 1 month and 18 days, or $86.40.

In the second instance his gain would be $100, less interest on $1600 for 1 month and 18 days, or $87.20.

In the first instance he pays for the goods and receives his profit at the end of the 3 months. In the second instance the transaction is not completed and profit realized until the maturity of the note.

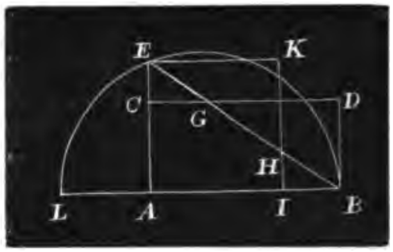
B. F. Burleson estimates *true* discount and does not count the days of grace. The *Proposer* neglects the days of grace.

37.—Proposed by B. F. BURLESON, Oneida Castle, Oneida County, New York.

A parsimonious farmer has a board $6\frac{1}{4}$ feet long and 2 feet wide, which he wishes to cut and form into a square without any waste of lumber. How must he cut the board and arrange the pieces?

Solution by BENJAMIN HEADLEY, Dillsborough, Dearborn Co., Indiana; GEORGE H. LELAND, Windsor, Windsor Co., Vermont; V. WEBSTER HEATH, Rodman, Jefferson Co., N. Y.; and John I. Clark, Moran, Clinton Co., Indiana.

Let ABDC represent the board. Produce AB till AL = AC, and on LB as a diameter describe a semicircle, and produce AC to E; then will AE = the side of the square which is equivalent to ABDC. Describe a square on AE as a side, and draw EB. Now, the three pieces ACGHI, HIB and GDB of the board are respectively equal to the three pieces ACGHI, ECG and EKH of the square.



Also answered in an ingenious manner by the *Proposer*, *Henry Heaton*, Prof. *Matz* and *K. S. Putnam*.

38.—Proposed by Prof. EDWARD BROOKS, M. A., Ph. D., Principal Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

A man gave me his note for \$500, payable in 8 years, interest at 6 per cent.; if he pays the interest annually, to what rate is this equivalent if the same amount of interest had been paid at the end of the time?

Solution by the PROPOSER; WILLIAM WILEY, Detroit, Wayne Co., Mich.; and F. P. MATZ, M. E., M. S., Professor of Higher Mathematics and Astronomy, King's Mountain High School, King's Mountain, Cleaveland Co., North Carolina.

Interest for 8 years = \$30 × 8 = \$240; interest on interest = \$30 × .06 = \$1.80; sum of periods = 28; \$1.80 × 28 = \$50.40; amount of interest = \$240 + \$50.40 = \$290.40; one year's interest = \$290.40 ÷ 8 = \$36.30; rate = \$36.30 ÷ 500 = $7\frac{13}{50}$ per cent.

Solved in a similar manner by *E. B. Seitz* and *K. S. Putnam*. *B. F. Burleson* and *John I. Clark* compute *compound* interest.

39.—Proposed by W. B. BATES, Earlville, LaSalle County, Illinois.

If Dr. A kills 3 patients out of 7; Dr. B, 4 out of 13; and Dr. C, 5 out of 19, what chance has a sick man for his life who employs all three of these doctors at the same time?

Solution by THEO. L. DELAND, U. S. Treasury Department, Washington, D. C.; SYLVESTER ROBINS, North Branch Depot, Somerset Co., N. J.; and WALTER S. NICHOLS, New York, N. Y.

If Dr. A kills 3 out of 7, $\frac{4}{7}$ remain; if Dr. B kills 4 out of 13, $\frac{9}{13}$ remain; if Dr. C kills 5 out of 19, $\frac{14}{19}$ remain. Then chance of life after all have "dosed" him is

$$\tfrac{4}{7}\times\tfrac{9}{13}\times\tfrac{14}{19}=\tfrac{72}{247}$$

Answered in like manner by Prof. *Matz*, *Henry Heaton* and *K. S. Putnam*.

40.—Proposed by J. G. WALTON, Covington, Kenton County, Kentucky.

Two circles have the same center. If a chord, length $2a$, be drawn across the larger circle, it will just touch the circumference of the smaller. Required, the area of the ring.

Solution by E. J. EDMUNDS, B. S., New Orleans, Orleans Co., La.; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and O. D. OATHOUT, Read, Clayton Co., Iowa.

Let R and r be the radii of the circles. Then $R^2-r^2=a^2$, and area of ring $=\pi(R^2-r^2)=\pi a^2$.

Substantially the same were the solutions by Messrs. *Albert, Burleson, Clark, DeLand, Heath, Heal, Martin, Matz, Norton, Putnam, Robins, Seitz, Shaw* and *Wiley*.

41.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

Find x and y from the equations

$$xy+(x^2+y^2)(1+xy+x^2y^2+x^3y+xy^3)=87,\quad xy(x^2+y^2)(x^2+xy+y^2)(x^2+y^2+xy+xy^3+x^3y)=1190.$$

I.—Solution by E. P. NORTON, Allen, Hillsdale Co., Mich.

The given equations may be put in the following form:

$$(xy+x^2+y^2)+xy(x^2+y^2)+xy(x^2+y^2)(xy+x^2+y^2)=87 \ldots\ldots(3),$$

$$xy(x^2+y^2)(xy+x^2+y^2)^2+x^2y^2(x^2+y^2)^2(xy+x^2+y^2)=1190 \ldots\ldots(4).$$

Put $v=xy+x^2+y^2$, and $w=xy(x^2+y^2)$; then we have $v+w+vw=87$......(5), $vw(v+w)=1190$......(6).

From (5) and (6) we readily find $v=7$ and $w=10$, or $xy+x^2+y^2=7$......(7) and $xy(x^2+y^2)=10$....(8); whence $x=2$ and $y=1$.

II.—Solution by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

From (1)

$$x^2+xy+y^2+xy(x^2+y^2)+xy(x^2+y^2)(x^2+xy+y^2)=87 \ldots\ldots(3);$$

hence we have the *sum* and *product* of the two expressions

$$x^2+xy+y^2+xy(x^2+y^2) \text{ and } xy(x^2+y^2)(x^2+xy+y^2),$$

whence

$$x^2+xy+y^2+xy(x^2+y^2)=17,\quad xy(x^2+y^2)(x^2+xy+y^2)=70.$$

Here we have also a sum and product,

whence $x^2+xy+y^2=7, \quad xy(x^2+y^2)=10.$

In a similar manner we find

$$x^2+y^2=5, \quad xy=2;$$

whence $x=2$ and $y=1$.

Good solutions given by Messrs. *Burleson*, *Leland*, *Matz*, *Putnam*, *Robins* and *Seitz*.

42.—Proposed by Benjamin Headley, Dillsborough, Dearborn County, Indiana.

Cut a board 3 feet wide and 7 feet long into three pieces, so that they will make a square.

[See solution of 37.]

43.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Two men. A and B, plow a rectangular field a rods long and b rods wide, A going ahead all the time. What part of the field does each plow, estimating n furrows to the rod?

Solution by Enoch Beery Seitz, Greenville, Darke County, Ohio.

Conceive the field to be divided into squares, the side of each of which is equal to the width of a furrow; there will be abn^2 squares. At each corner after the first A gains two squares on B.

1.—When bn is an even number, and the first two furrows are plowed along the side of the field, a being greater than b, there will be $\frac{1}{2}bn$ rounds, and A will gain $2(bn-1)$ squares;

$$\therefore \frac{1}{2}+\frac{bn-1}{abn^2}=\frac{1}{2}+\frac{1}{an}-\frac{1}{abn^2}=\text{A's part, and } \frac{1}{2}-\frac{bn-1}{abn^2}=\frac{1}{2}-\frac{1}{an}+\frac{1}{abn^2}=\text{B's part.}$$

2.—When bn is an even number, and the first two furrows are plowed across the end of the field, there will be $\frac{1}{2}bn$ rounds, and A will plow one square from the end of B's last furrow; hence A will gain $2bn$ squares;

$$\therefore \frac{1}{2}+\frac{bn}{abn^2}=\frac{1}{2}+\frac{1}{an}=\text{A's part, and } \frac{1}{2}-\frac{bn}{abn^2}=\frac{1}{2}-\frac{1}{an}=\text{B's part.}$$

3.—When bn is an odd number, and the first two furrows are plowed along the side of the field, there will be $\frac{1}{2}(bn-1)$ rounds, and a furrow of $(an-bn+3)$ squares, which A will plow; hence A will gain $2(bn-2)+an-bn+3=(an+bn-1)$ squares, which will also be the gain when the first two furrows are plowed across the end of the field, as may easily be shown;

$$\therefore \frac{1}{2}+\frac{an+bn-1}{2abn^2}=\frac{1}{2}+\frac{1}{2an}+\frac{1}{2bn}-\frac{1}{2abn^2}=\text{A's part,}$$

and

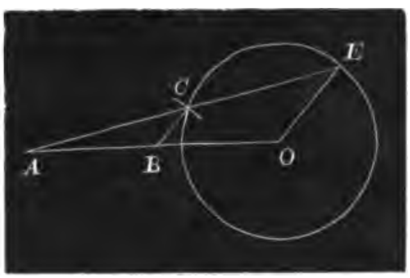
$$\frac{1}{2}-\frac{an+bn-1}{2abn^2}=\frac{1}{2}-\frac{1}{2an}-\frac{1}{2bn}+\frac{1}{2abn^2}=\text{B's part.}$$

Solved in a similar manner by *Frank Albert*, *B. F. Burleson*, *K. S. Putnam* and *Walter Siverly*.

44.—Proposed by Winfield V. Jeffries, Instructor in Mathematics, Vermillion Institute, Hayesville, Ashland Co., O.

From a given point without a circle, to draw a secant that shall be bisected by the circumference.

Solution by E. B. Seitz, Greenville, Darke Co., Ohio; and William Hoover, Mathematical Editor *Wittenberger*, Bellefontaine, Logan Co., Ohio.



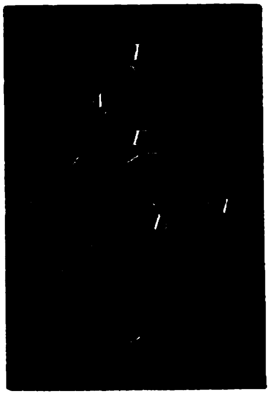
Let A be the given point, O the center of the circle, and B the middle point of AO. With B as a center and a radius equal to half the given radius describe an arc cutting the given circumference in C. Draw OE parallel to BC, meeting AC produced in E. Now since AO = 2AB, we have by similar triangles OE = 2BC = the given radius, and EC = AC; hence E is on the given circumference, and the secant AE is bisected at C.

Good solutions received from Messrs. *Albert*, *Baker*, *Burleson*, *Edmunds*, *Heath*, *McAdam*, *Putnam*, *Siverly* and *Wiley*.

45.—Proposed by Mrs. Anna T. Snyder, Orange, Fayette Co., Indiana.

A father having 640 acres of land in a circle gives to each of his six sons a circular farm touching the circumference of the circle and also the circumferences of the farms of two of his brothers, the six farms being equal in size. Required the number of acres in the farm of each of the sons, the number remaining to the father, the number remaining in the center, and the number in each of the other six equal pieces.

Solution by K. S. Putnam, Rome, Oneida Co., N. Y.; B. F. Burleson, Oneida Castle, Oneida Co., N. Y.; and Marcus Baker, U. S. Coast Survey Office, Washington, D. C.



Let radius $CD = R$, radius of each smaller circle $= r$. The lines joining the centers of the inscribed circles will be the sides of a regular hexagon, center at C; therefore $CB = BA = 2r$, $CD = 3r$ and $r = \frac{1}{3}R$.

Similar surfaces are as squares of similar parts, therefore each inscribed circle $= \frac{1}{9}$ of the father's farm, $= 71\frac{1}{9}$ acres. The part remaining to the father $= \frac{1}{3}$ of $640 = 213\frac{1}{3}$ acres.

The area of the sector $ECD = \frac{1}{6}\pi R^2$. If from this we take the equilateral triangle $ABC = r^2\sqrt{3} = \frac{1}{9}R^2\sqrt{3}$, and the two sectors EAF and DBF, each $\frac{1}{3}$ of an inscribed circle and both $= \frac{2}{3}\pi r^2 = \frac{2}{27}\pi R^2$, there will remain

$$\tfrac{1}{6}\pi R^2 - \tfrac{1}{9}R^2\sqrt{3} - \tfrac{2}{27}\pi R^2 = \frac{5\pi R^2 - 6R^2\sqrt{3}}{54}.$$

Whole circle $= \pi R^2 = 640$ acres;

$$\therefore \quad \pi R^2 : \frac{5\pi R^2 - 6R^2\sqrt{3}}{54} :: 640 : \text{area EFD};$$

whence area $EFD = \dfrac{320(5\pi - 6\sqrt{3})}{27\pi} = 20.0536+$ acres.

$213\frac{1}{3} - 6(\text{area EFD}) = 93.0114+$ acres, the part in the center.

Nearly thus were the solutions by Messrs. *Martin*, *McAdam*, *Oathout*, *Seitz*, *Wiley* and the *Proposer*.

46.—Proposed by J. F. W. Scheffer, Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin Co., Pa.

The sides of a plane triangle are proportional to the roots of the cubic equation

$$x^3 - ax^2 + bx - c = 0.$$

Find the sum of the cosines of the angles of the triangle.

Solution by W. E. Heal, Wheeling, Delaware County, Indiana.

Let p, q, r denote the roots of the equation; kp, kq, kr the sides of the triangle; and A, B, C the angles respectively opposite.

Then

$$\cos A = \frac{q^2 + r^2 - p^2}{2qr}, \quad \cos B = \frac{p^2 + r^2 - q^2}{2pr}, \quad \cos C = \frac{p^2 + q^2 - r^2}{2pq};$$

$$\therefore \quad \cos A + \cos B + \cos C = \frac{(pq^2 + pr^2 + p^2q + qr^2 + p^2r + q^2r) - (p^3 + q^3 + r^3)}{2pqr}.$$

But from the Theory of Equations

$$p + q + r = a, \ \ pq + qr + pr = b, \ \ pqr = c; \ \ \therefore pq^2 + pr^2 + p^2q + qr^2 + p^2r + q^2r = ab - 3c,$$

$$p^3 + q^3 + r^3 = a^3 - 3ab + 3c \text{ and } \cos A + \cos B + \cos C = \frac{4ab - a^3 - 6c}{2c}.$$

Answered in a similar manner by the *Proposer* and Messrs. *Albert*, *Burleson*, *Matz*, *McAdam*, *Putnam*, *Seitz* and ——.

47.—Proposed by Daniel Kirkwood, LL.D., Professor of Mathematics, Indiana State University, Bloomington, Monroe Co., Ind.

The number of diagonals that can be drawn in a certain polygon is equal to seven times the number of sides. How many sides has the polygon?

Solution by ——; W. E. Heal; K. S. Putnam; and William Wiley.

The number of diagonals that can be drawn from any angle of a polygon of n sides is $n - 3$; hence the number of diagonals that can be drawn from the n angles is $n(n-3)$; but each diagonal of the polygon will thus have been drawn twice. The number of diagonals is therefore $\frac{n(n-3)}{2}$. In the problem on hand we have, therefore, the equation $\frac{n(n-3)}{2} = 7n$, whence $n = 17$.

Solved also by Messrs. *Albert*, *Baker*, *Clark*, *Edmunds*, *DeLand*, *Heal*, *Matz*, *McAdam*, *Merrill*, *Norton*, *Oathout*, *Putnam*, *Robins*, *Seitz*, *Trowbridge* and Mrs. *Anna T. Snyder*.

48.—Proposed by Joseph Ficklin, M. A., Ph. D., Professor of Mathematics and Astronomy, University of the State of Missouri, Columbia, Boone Co., Mo.

Find the radius of a sphere that shall circumscribe four equal spheres which touch each other.

Solution by the Proposer; V. Webster Heath; Prof. F. P. Matz, M. E., M. S.; Marcus Baker; B. F. Burleson; and E. B. Seitz.

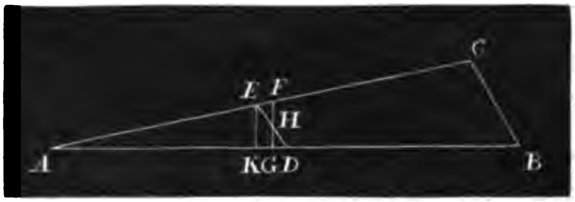
Suppose the radius of the given sphere to be a. Form a tetraedron by joining the centers of the four spheres. The edge of this tetraedron will be $2a$. Again, $e = \frac{2}{3}R\sqrt{6}$ expresses the relation between the radius of a sphere and the edge of the inscribed regular tetraedron; therefore $2a = \frac{2}{3}R\sqrt{6}$; whence,

$R = \frac{1}{2}a\sqrt{6}$. This is the radius of the sphere whose surface passes through the centers of the four given spheres; therefore $\frac{1}{2}a\sqrt{6} + a = a(1 + \frac{1}{2}\sqrt{6})$ is the radius of the sphere which circumscribes them.

49.—Proposed by the late Dr. S. F. BACHELDER, South Boston, Massachusetts.

P and Q owned a triangular piece of land, the sides of which were 20, 18 and 4 chains. The longest side bordered on a road which ran east and west. They agreed to divide it by a line parallel to the shortest side so that Q, who took the resulting trapezoid, should have $\frac{3}{4}$ of the land. A surveyor located the line; then P, setting a stake at the middle of it, proposed that a new line should be run north and south through that point, so as to give both a square corner on the road. Q assented; did he gain or lose, in area, by the change of line, and how much?

Solution by B. F. BURLESON, Oneida Castle, Oneida Co., N. Y.; and E. B. SEITZ, Greenville, Darke Co., O.



Let ABC represent the triangle, and $a = 4$, $b = 18$, and $c = 20$ be the sides. Let ED be the surveyor's located division line. The area of the triangle ABC $= 8\sqrt{(119)}$ square chains. Therefore P's share $= \frac{3}{4}\sqrt{(119)}$ square chains. The triangles ABC and ADE being similar we have $4 : 1 : : (4)^2 : (ED)^2$; $\therefore$ ED $= 2$ chains. In a similar manner we find AD $= 10$ chains. Through the middle point H of the line ED draw FG perpendicular to AB, the points F and G being in the lines AC and AB respectively. In the right-angled triangle HGD we have given the hypothenuse HD $= 1$ chain, and the angle GDH, $=$ the angle ABC, to find the remaining parts.

By Trigonometry, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac} = \frac{23}{40}$, and GD $= \frac{23}{40}$ of a chain. Whence GH $= \frac{3}{40}\sqrt{(119)}$, and AG $=$ AD $-$ GD $= \frac{377}{40}$ chains. From E let fall the perpendicular EK upon AB. Then because the triangles EKD and HGD are similar, and because ED $= 2$HD, we have KD $= 2$GD $= \frac{23}{20}$, EK $= 2$GH $= \frac{3}{20}\sqrt{(119)}$, and AK $=$ AD $-$ KD $= \frac{177}{20}$ chains.

Therefore the area of the triangle AKE $= \frac{177}{20} \times \frac{3}{40}\sqrt{(119)} = \frac{531}{800}\sqrt{(119)}$ square chains. Now because the triangle AKE is similar to the triangle AGF, we have $(AK)^2 : (AG)^2 : : \triangle AKE : \triangle AGF$;
that is, $(\frac{177}{20})^2 : (\frac{377}{40})^2 : : \frac{531}{800}\sqrt{(119)} : \frac{142129}{188800}\sqrt{(119)}$ square chains $=$ area of triangle AGF.

Therefore by the change of division lines Q loses $\frac{3}{4}\sqrt{(119)} - \frac{142129}{188800}\sqrt{(119)} = \frac{529}{188800}\sqrt{(119)} =$ 0.0305651 + of a square chain, $=$ 0.48904 + of a square rod.

Good solutions received from Messrs. *Albert*, *Baker*, *Ludwig*, *Matz*, *Nichols*, *Oathout*, *Putnam*, *Robins* and *Wiley*.

50.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In any plane triangle ABC the angles B and C are bisected, the bisectors meeting the sides in B′ and C′ respectively. Join B′ and C′ and from any point P in B′C′ let perpendiculars p_a, p_b and p_c, fall upon the sides a, b and c respectively; prove that $p_a = p_b + p_c$.

Solution by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

From B′ and C′ draw parallels P_c and P_b to p_c and p_b respectively; also, from B′ and C′ draw parallels P_a and $P_{a'}$ to $p_{a'}$, and suppose $P_a < P_{a'}$; then let fall a perpendicular from B′ to $P_{a'}$; also let m and n be the parts into which the line B′C′ is divided; then by similar triangles

$$m + n : P_c : : n : p_c = \frac{nP_c}{m+n} \;\ldots\; (1), \text{ and } m+n : P_b : : m : p_b = \frac{mP_b}{m+n} \;\ldots\ldots (2);$$

also
$$m + n : P_{a'} - P_a : : m : p_a - P_a = \frac{m(P_{a'} - P_a)}{m+n}, \text{ whence } p_a = \frac{mP_{a'} + nP_a}{m+n} \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

But
$$P_{a'} = P_b \text{ and } P_a = P_c, \text{ hence } p_a = \frac{mP_b + nP_c}{m+n}, = p_b + p_c \text{ from (1) and (2).}$$

Excellent solutions sent by *E. B. Seitz*, *K. S. Putnam* and *William Hoover*.

51.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

There is a series of right-angled triangles whose legs differ by unity only. Calling the one whose sides are 3, 4, 5 the *first* triangle, it is required to find general expressions for the sides of the nth triangle, and compute the sides of the 100th triangle.

Solution by the PROPOSER.

Let $\frac{1}{2}(x_n - 1) = p_n =$ perpendicular, $\frac{1}{2}(x_n + 1) = b_n =$ base, and $y_n = h_n =$ hypothenuse of the nth triangle.

Then $\quad \frac{1}{4}(x_n - 1)^2 + \frac{1}{4}(x_n + 1)^2 = y_n^2$; whence $x_n^2 - 2y_n^2 = -1$ ....(1),

or $\quad (x_n - y_n\sqrt{2})(x_n + y_n\sqrt{2}) = -1$ ....(2).

We also have $(x_0 - y_0\sqrt{2})(x_0 + y_0\sqrt{2}) = -1$, and $(x_0 - y_0\sqrt{2})^{2n+1}(x_0 + y_0\sqrt{2})^{2n+1} = -1$ ....(3), where n may be 0, 1, 2, 3, &c.

Assuming $x_n - y_n\sqrt{2} = (x_0 - y_0\sqrt{2})^{2n+1}$ and $x_n + y_n\sqrt{2} = (x_0 + y_0\sqrt{2})^{2n+1}$, which we are at liberty to do, we find

$$x_n = \frac{1}{2}\left[(x_0 + y_0\sqrt{2})^{2n+1} + (x_0 - y_0\sqrt{2})^{2n+1}\right], \quad y_n = \frac{1}{2\sqrt{2}}\left[(x_0 + y_0\sqrt{2})^{2n+1} - (x_0 - y_0\sqrt{2})^{2n+1}\right].$$

But it is easily seen that $x_0 = 1$, $y_0 = 1$;

$$\therefore \quad x_n = \frac{1}{2}\left[(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1}\right], \quad y_n = \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}\right],$$

and the sides of the nth triangle are

$$p_n = \frac{1}{2}(x_n - 1) = \frac{1}{4}\left[(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1} - 2\right], \quad b_n = \frac{1}{2}(x_n + 1) =$$

$$\frac{1}{4}\left[(\sqrt{2}+1)^{2n+1} - (\sqrt{2}-1)^{2n+1} + 2\right], \quad h_n = y_n = \frac{1}{4\sqrt{2}}\left[(\sqrt{2}+1)^{2n+1} + (\sqrt{2}-1)^{2n+1}\right].$$

The operation of involution is very tedious, except when n is a small number.

We have from (1), $x_n = \sqrt{(2y_n^2 - 1)}$. When x_n and y_n are very large numbers, the 1 under the radical may be omitted without sensible error, and we have, for the superior limit of their ratio, $\frac{x_n}{y_n} = \sqrt{2}$; and the values of x_n and y_n are the numerators and denominators of the odd convergents to the square root of 2 expanded as a continued fraction.

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \text{etc.}}}};$$

the successive convergents are $\frac{1}{1}$, $\frac{3}{2}$, $\frac{7}{5}$, $\frac{17}{12}$, $\frac{41}{29}$, $\frac{99}{70}$, $\frac{239}{169}$, etc.

Writing $\quad \frac{q_1}{r_1}, \frac{q_3}{r_3}, \frac{q_5}{r_5}, \frac{q_7}{r_7}, \ldots \frac{q_{2n+1}}{r_{2n+1}}$ for the odd convergents, we have

$h_n = r_{2n+1}$, $b_n = \frac{1}{2}(q_{2n+1} + 1)$, $p_n = \frac{1}{2}(q_{2n+1} - 1)$; $\quad \therefore \quad h_{100} = r_{201}$, $b_{100} = \frac{1}{2}(q_{201} + 1)$, $p_{100} = \frac{1}{2}(q_{201} - 1)$.

When $n = 1$, $h_1 = r_3 = 5$, $b_1 = 4$, $p_1 = 3$, the sides of the *first* triangle.

The sides of the second triangle are	20, 21, 29;
of the third,	119, 120, 169;
of the fourth,	696, 697, 985;
of the fifth,	4058, 4059, 5741;
of the sixth,	23657, 23658, 33457; etc., etc.

In finding the sides of very large triangles it will be convenient to employ the formulas

$$\frac{q_{2c}}{r_{2c}} = \frac{2q_c^2 + 1}{2q_c r_c} \quad \ldots (4), \quad \frac{q_{3c}}{r_{3c}} = \frac{q_c(4q_c^2 + 3)}{r_c(4q_c^2 + 1)} \quad \ldots\ldots(5),$$

$$\frac{q_{2m+3}}{r_{2m+3}} = \frac{4r_{2m+1} + 3q_{2m+1}}{3r_{2m+1} + 2q_{2m+1}} \quad \ldots (6), \quad \frac{q_{2m+1}}{r_{2m+1}} = \frac{3q_{2m+3} - 4r_{2m+3}}{3r_{2m+3} - 2q_{2m+3}} \quad \ldots\ldots (7),$$

where c may be any odd number, and m may be any number.

Take $c = 7$, then by (5) $\frac{q_{21}}{r_{21}} = \frac{54608393}{38613965}$, and by taking $m = 10$ (6) gives $\frac{q_{23}}{r_{23}} = \frac{318281039}{225058681}$.

Take $c = 23$, then $\frac{q_{69}}{r_{69}} = \frac{128971066941642015967892393}{91196316011299234022705885}$, and by taking $m = 33$ (7) gives

$$\frac{q_{67}}{r_{67}} = \frac{22127936779729111812858639}{15646814150613670132332869}.$$

Now take $c = 67$ and we have by (5)

$q_{201} = 433393862972275766610959594585726143280304055873989306921633883984677691985393$,

$r_{201} = 30645573943282956180057972969833245887630954508753693529117371084705767728665$.

$\therefore \quad p_{100} = 216696931486137888305479797292863071640152027686994653460816919923388459926 96$,

$b_{100} = 21669693148613788830547979729286307164015202768699465346081691992338845992697$,

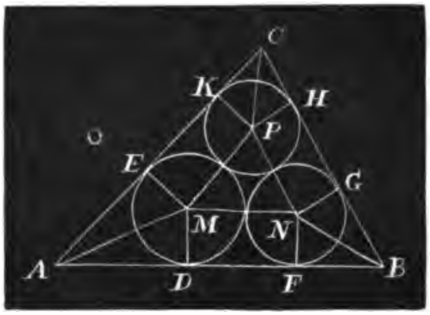
$h_{100} = 30645573943232956180057972969833245887630954508753693529117371074705767728665$.

Mr. *Putnam*, by an elegant but laborious method, computes the sides of the 100th triangle, but does not find the general values of the sides of the nth triangle. Messrs. *Albert*, *Hoover*, *Hart*, *Robins* and *Seitz* give general expressions for the sides of the nth triangle, but do not compute the sides of the 100th triangle.

52.—Proposed by JAMES MCLAUGHLIN, Mantorville, Dodge County, Minnesota.

Three circles, radii a, b, c, are drawn in a triangle, each circle touching the other two and two sides of the triangle. Find the sides of the triangle.

Solution by E. B. SEITZ, Greenville, Darke County, Ohio.



Let ABC be the triangle, M, N, P the centers of the three circles. Let MD = ME = a, NF = NG = b, PH = PK = c, $a+b+c=s$, $\angle$NMP $=\theta$, $\angle$NMD $=\varphi$, $\angle$PME $=\psi$, $\angle$DME $=\omega$. Then DF $=2\sqrt{(ab)}$, EK $=2\sqrt{(ac)}$, GH $=2\sqrt{(bc)}$, $\sin\varphi=\dfrac{2\sqrt{(ab)}}{a+b}$, $\cos\varphi=\dfrac{a-b}{a+b}$, $\sin\psi=\dfrac{2\sqrt{(ac)}}{a+c}$, $\cos\psi=\dfrac{a-c}{a+c}$, $\tan\frac{1}{2}\theta=\sqrt{\left(\dfrac{bc}{as}\right)}$, $\tan\frac{1}{2}\varphi=\dfrac{1-\cos\varphi}{\sin\varphi}=\sqrt{\left(\dfrac{b}{a}\right)}$, $\tan\frac{1}{2}\psi=\dfrac{1-\cos\psi}{\sin\psi}=\sqrt{\left(\dfrac{c}{a}\right)}$, and $\frac{1}{2}\omega=\pi-\frac{1}{2}(\theta+\varphi+\psi)$; whence $\tan\frac{1}{2}\varphi=-\tan\frac{1}{2}(\theta+\varphi+\psi)$,

$$=\frac{\tan\frac{1}{2}\theta+\tan\frac{1}{2}\varphi+\tan\frac{1}{2}\psi-\tan\frac{1}{2}\theta\tan\frac{1}{2}\varphi\tan\frac{1}{2}\psi}{\tan\frac{1}{2}\theta\tan\frac{1}{2}\varphi+\tan\frac{1}{2}\theta\tan\frac{1}{2}\psi+\tan\frac{1}{2}\varphi\tan\frac{1}{2}\psi-1}$$

$$=\frac{a\sqrt{(abc)}+a\sqrt{(abs)}+a\sqrt{(acs)}-bc\sqrt{a}}{ab\sqrt{c}+ac\sqrt{b}+a\sqrt{(bcs)}-a^2\sqrt{s}}$$

$$\therefore\quad \text{AD}=a\tan\tfrac{1}{2}\omega=\frac{a\sqrt{(abc)}+a\sqrt{(abs)}+a\sqrt{(acs)}-bc\sqrt{a}}{b\sqrt{c}+c\sqrt{b}+\sqrt{(bcs)}-a\sqrt{s}}.$$

Similarly we have
$$\text{BF}=\frac{b\sqrt{(abc)}+b\sqrt{(abs)}+b\sqrt{(bcs)}-ac\sqrt{b}}{a\sqrt{c}+c\sqrt{a}+\sqrt{(acs)}-b\sqrt{s}}.$$

$$\therefore\quad \text{AB}=\text{AD}+\text{DF}+\text{FB}=\frac{(a+b)^2\left[c\sqrt{(ab)}-b\sqrt{(ac)}-a\sqrt{(bc)}+c\left\{\sqrt{(as)}+\sqrt{(bs)}+\sqrt{(cs)}\right\}-\sqrt{(abcs)}+cs\right]}{\left\{a\sqrt{c}+c\sqrt{a}+\sqrt{(acs)}-b\sqrt{s}\right\}\left\{b\sqrt{c}+c\sqrt{b}+\sqrt{(bcs)}-a\sqrt{s}\right\}}.$$

By symmetry,
$$\text{AC}=\frac{(a+c)^2\left[b\sqrt{(ac)}-c\sqrt{(ab)}-a\sqrt{(bc)}+b\left\{\sqrt{(as)}+\sqrt{(bs)}+\sqrt{(cs)}\right\}-\sqrt{(abcs)}+bs\right]}{\left\{a\sqrt{b}+b\sqrt{a}+\sqrt{(abs)}-c\sqrt{s}\right\}\left\{b\sqrt{c}+c\sqrt{b}+\sqrt{(bcs)}-a\sqrt{s}\right\}},$$

and
$$\text{BC}=\frac{(b+c)^2\left[a\sqrt{(bc)}-c\sqrt{(ab)}-b\sqrt{(bc)}+a\left\{\sqrt{(as)}+\sqrt{(bs)}+\sqrt{(cs)}\right\}-\sqrt{(abcs)}+as\right]}{\left\{a\sqrt{b}+b\sqrt{a}+\sqrt{(abs)}-c\sqrt{s}\right\}\left\{a\sqrt{c}+c\sqrt{a}+\sqrt{(acs)}-b\sqrt{s}\right\}}.$$

Unfinished solutions were received from Messrs. *Baker*, *Burleson* and *Matz*.

53.—Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

The towns C and D are 40 miles apart. A set out from C to travel to D at the same time that B started from D to go to C. A *overtook* a drove of sheep moving forward at the rate of 2 miles an hour just 10 minutes *after* he crossed a creek known to be 10 miles from C; B arrived at C 3 hours and 55 minutes after he *met* the same drove. B was *overtaken* by an express proceeding at the rate of 5 miles an hour just 5 minutes *before* he came to an inn 11 miles from D; A arrived at D 8 hours after he *met* the express. Required the hourly speed of A and B.

[Owing to a clerical error, the distance of the inn from B and the time A traveled after he *met* the express were printed wrong in No. 2.]

Solution by DUNLAP J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania.

Let $x=$ A's hourly speed, and $y=$ B's. Then $10+\dfrac{x}{6}=$ distance A traveled till he *overtook* the drove.

$3\frac{11}{12}y=$ distance B traveled after he *met* the drove......(a).

$\dfrac{47}{12}y-10-\dfrac{x}{6}=$ distance drove goes from A *overtook* till B *met* it. $\dfrac{47y-120-2x}{24}=$ *time* from A *overtook* till B *met* the drove. $\dfrac{(47y-120-2x)y}{24}=$ distance B traveled in that time............(b).

$\dfrac{10+\frac{1}{6}x}{x}=$ *time* A and B had traveled before A *overtook* the drove. $\dfrac{(60+x)y}{6x}=$ distance B traveled in that time......(c.)

$$\therefore\quad \text{B's whole distance}=\frac{47y}{12}+\frac{47y^2-120y-2xy}{24}+\frac{60y+xy}{6x}=40.$$

Reducing,
$$47xy^2-2x^2y-22xy-960x+240y=0\text{...}(1).$$

By second condition of the question,

$11-\frac{y}{12}=$ distance of B from D when express *overtook* him. $8x=$ distance A traveled after he met the express......(*d*). $8x-11+\frac{y}{12}=$ distance express traveled from it *overtook* B till it *met* A.

$\frac{96x-132+y}{60}=$ *time* from express overtook B till it *met* A. $\frac{96x^2-132x+xy}{60}=$ distance A goes from express *overtook* B till it met A......(*e*). $\frac{132-y}{12y}=$ time A and B had traveled when express *overtook* B.

$\frac{132x-xy}{12y}=$ distance A traveled before express overtook B......(*f*).

$$\therefore \quad \text{A's distance}=8x+\frac{96x^2-132x+xy}{60}+\frac{132x-xy}{12y}=40.$$

Reducing, $xy^2+96x^2y+343xy+660x-2400y=0$ ....(2).

From (1) and (2), by elimination, we find

$533xy+1514x^2+5246x-35760=0$.................... (3), $471y^2+76xy+123y-8940=0$..... ....(4).

Finding y in (3) $=\frac{35760-5246x-1514x^2}{533x}$ and substituting in (4),

$$3417098x^4+24060473x^3-132461181x^2-585140760x+2021155200=0.$$

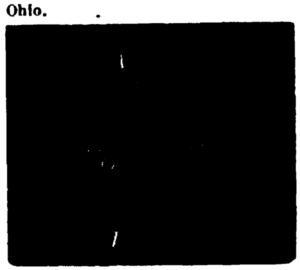
Solving for one value, we find $x=3$, and then $y=4$.

This problem was solved in an elegant manner by Messrs. *Burleson*, *Putnam* and *Seitz*.

54.—Proposed by E. J. Rowan, Shawnee, Perry County, Ohio.

What length of line, fastened to a point in the circumference of a circle whose area is one acre, will allow an animal to graze upon just one acre outside of the circle?

Solution by Enoch Berry Seitz, Greenville, Darke County, Ohio.



Let C be the point to which the line is fastened, A and B the points in the given circumference, to which the animal can graze, and O the center of the given circle Let $OA=a=4\sqrt{\left(\frac{10}{\pi}\right)}$, the radius of the given circle, and $\angle ACO=\theta$. Then we have $AC=2a\cos\theta$, the length of the line, and the area common to the given circle and the circle upon which the animal can graze $=a^2(\pi+2\theta\cos 2\theta-\sin 2\theta)$; therefore the area upon which the animal can graze outside of the circle is

$$4\pi a^2\cos^2\theta-a^2(\pi+2\theta\cos 2\theta-\sin 2\theta)=\pi a^2, \text{ whence } 2\theta-\tan 2\theta=2\pi.$$

Solving this equation by the method of Double Position, we find $\theta=51°16'24''$. $\therefore$ $AC=2a\cos\theta=8.92926$ rods.

William Hoover used Calculus in his solution. Prof. *F. P. Matz* furnished a good solution without numerical result.

55.—Prize Problem. Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Given

$y=x+ax^2+bx^3+cx^4+dx^5+ex^6+fx^7+gx^8+hx^9+ix^{10}+jx^{11}+kx^{12}+lx^{13}+mx^{14}+nx^{15}+px^{16}+\&c$ (1),

and

$x=y+Ay^2+By^3+Cy^4+Dy^5+Ey^6+Fy^7+Gy^8+Hy^9+Iy^{10}+Jy^{11}+Ky^{12}+Ly^{13}+My^{14}+Ny^{15}+Py^{16}+\&c.$..(2);

to find A, B, C, D, E, F, G, H, I, J, K, L, M, N, P in terms of $a, b, c, d, e, f, g, h, i, j, k, l, m, n, p$.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Substituting the value of y from (1) in (2), and collecting the terms having like powers of x, we have

$x=x+(A+a)x^2+(2aA+B+b)x^3+[(a^2+2b)A+3aB+C+c]x^4+[(2ab+2c)A+(3a^2+3b)B+4aC+D+d]x^5$
$+[(2ac+b^2+2d)A+(a^3+6ab+3c)B+(6a^2+4b)C+5aD+E+e]x^6+[(2ad+2bc+2e)A+(3a^2b+6ac+3b^2+3d)B$
$+(4a^3+12ab+4c)C+(10a^2+5b)D+6aE+F+f]x^7+[(2ae+2bd+c^2+2f)A+(3a^2c+3ab^2+6ad+6bc+3e)B+(a^4$
$+12a^2b+12ac+6b^2+4d)C+(10a^3+20ab+5c)D+(15a^2+6b)E+7aF+G+g]x^8+[(2af+2be+2cd+2g)A+(3a^2d$
$+6abc+6ae+b^3+6bd+3c^2+3f)B+(4a^3b+12a^2c+12ab^2+12ad+12bc+4e)C+(5a^4+30a^2b+20ac+10b^2+5d)D$
$+(20a^3+30ab+6c)E+(21a^2+7b)F+8aG+H+h]x^9+[(2ag+2bf+2ce+d^2+2h)A+(3a^2e+6abd+3ac^2+6af$
$+3b^2c+6be+6cd+3g)B+(4a^3c+6a^2b^2+12a^2d+24abc+12ae+4b^3+12bd+6c^2+4f)C+(a^5+20a^3b+30a^2c+30ab^2$
$+20ad+20bc+5e)D+(15a^4+60a^2b+30ac+15b^2+6d)E+(35a^3+42ab+7c)F+(28a^2+8b)G+9aH+I+i]x^{10}$
$+\&c$ ....(3).

By the principle of Indeterminate Coefficients we have from (3) $A+a=0$, $2aA+B+b=0$, $(a^2+2b)A+3aB+C+c=0$, $(2ab+2c)A+(3a^2+3b)B+4aC+D+d=0$, &c.

From these equations we find

$A = -a, \quad B = 2a^2 - b, \quad C = -5a^3 + 5ab - c, \quad D = 14a^4 - 21a^2b + 6ac + 3b^2 - d,$

$E = -42a^5 + 84a^3b - 28a^2c - 28ab^2 + 7ad + 7bc - e,$

$F = 132a^6 - 330a^4b + 120a^3c + 180a^2b^2 - 36a^2d - 72abc - 12b^3 + 8ae + 8bd + 4c^2 - f,$

$G = -429a^7 + 1287a^5b - 495a^4c - 990a^3b^2 + 165a^3d + 165ab^3 + 495a^2bc - 45a^2e - 45ac^2 - 45b^2c - 90abd$
$+ 9af + 9be + 9cd - g,$

$H = 1430a^8 - 5005a^6b + 2002a^5c + 5005a^4b^2 - 715a^4d - 2860a^3bc - 1430a^2b^3 + 220a^3e + 330a^2c^2 + 660a^2bd$
$+ 660ab^2c + 55b^4 - 55a^2f - 55b^2d - 55bc^2 - 110abe - 110acd + 10ag + 10bf + 10ce + 5d^2 - h,$

$I = -4862a^9 + 19448a^7b - 8008a^6c - 24024a^5b^2 + 3003a^5d + 15015a^4bc + 10010a^3b^3 - 1001a^4e - 4004a^3bd$
$- 2002a^3c^2 - 6006a^2b^2c + 286a^3f + 286b^3c + 858a^2be + 858a^2cd + 858ab^2d + 858abc^2 - 66a^2g - 66ad^2$
$- 66b^2e - 132abf - 132ace - 132bcd - 22c^3 + 11ah + 11bg + 11cf + 11de - i,$

$J = 16796a^{10} - 75582a^8b + 31824a^7c + 111384a^6b^2 - 12376a^6d - 74256a^5bc - 61880a^4b^3 + 4368a^5e$
$+ 21840a^4bd + 10920a^4c^2 + 10920a^2b^4 + 43680a^3b^2c - 1365a^4f - 5460a^3be - 5460a^3cd - 5460ab^3c$
$- 8190a^2b^2d - 8190a^2bc^2 - 273b^5 + 364a^3g + 364b^3d + 364ac^3 + 1092a^2bf + 1092a^2ce + 1092ab^2e$
$+ 546a^2d^2 + 546b^2c^2 + 2184abcd - 78a^2h - 78b^2f - 78c^2d - 78bd^2 - 156abg - 156acf - 156ade$
$- 156bce + 12ai + 12bh + 12cg + 12df + 6e^2 - j,$

$K = -58786a^{11} + 293930a^9b - 125970a^8c - 503880a^7b^2 + 50388a^7d + 352716a^6bc + 352716a^5b^3 - 18564a^6e$
$- 111384a^5bd - 55692a^5c^2 - 278460a^4b^2c - 92820a^3b^4 + 6188a^5f + 6188ab^5 + 30940a^4be + 30940a^4cd$
$+ 61880a^3b^2d + 61880a^3bc^2 + 61880a^2b^3c - 1820a^4g - 1820b^4c - 7280a^3bf - 7280a^3ce - 7280ab^3d$
$- 3640a^3d^2 - 3640a^2c^3 - 10920a^2b^2e - 10920ab^2c^2 - 21840a^2bcd + 455a^3h + 455b^3e + 455bc^3$
$+ 1365a^2bg + 1365a^2cf + 1365a^2de + 1365ab^2f + 1365b^2cd + 1365ac^2d + 1365abd^2 + 2730abce$
$- 91a^2i - 91b^2g - 91c^2e - 91cd^2 - 91ae^2 - 182abh - 182acg - 182adf - 182bcf - 182bde + 13aj$
$+ 13bi + 13ch + 13dg + 13ef - k,$

$L = 208012a^{12} - 1144066a^{10}b + 497420a^9c + 2238390a^8b^2 - 203490a^8d - 1627920a^7bc - 1899240a^6b^3$
$+ 77520a^7e + 542640a^6bd + 271320a^6c^2 + 1627920a^5b^2c + 678300a^4b^4 - 27132a^6f - 162792a^5be$
$- 162792a^5cd - 406980a^4b^2d - 406980a^4bc^2 - 542640a^3b^3c - 81396a^2b^5 + 8568a^5g + 42840a^4bf$
$+ 42840a^4ce + 42840ab^4c + 21420a^4d^2 + 85680a^3b^2e + 85680a^2b^3d + 171360a^3bcd + 28560a^3c^3$
$+ 128520a^2b^2c^2 + 1428b^6 - 2380a^4h - 2380b^4d - 9520a^3bg - 9520a^3cf - 9520a^3de - 9520ab^3e - 9520abc^3$
$- 14280a^2b^2f - 14280a^2c^2d - 14280a^2bd^2 - 28560a^2bce - 28560ab^2cd - 4760b^3c^2 + 560a^3i + 560b^3f$
$+ 1680a^2bh + 1680a^2cg + 1680a^2df + 1680ab^2g + 1680b^2ce + 1680ac^2e + 1680bc^2d + 1680acd^2 + 840a^2e^2$
$+ 840b^2d^2 + 3360abcf + 3360abde + 140c^4 - 105a^2j - 105b^2h - 105c^2f - 105be^2 - 210abi - 210ach$
$- 210adg - 210aef - 210bcg - 210bdf - 210cde - 35d^3 + 14ak + 14bj + 14ci + 14dh + 14eg$
$+ 7f^2 - l,$

$M = -742900a^{13} + 4457400a^{11}b - 1961256a^{10}c - 9806280a^9b^2 + 817190a^9d + 7354710a^8bc + 9806280a^7b^3$
$- 319770a^8e - 2558160a^7bd - 1279080a^7c^2 - 8953560a^6b^2c - 4476780a^5b^4 + 116280a^7f + 813960a^6be$
$+ 813960a^6cd + 2441880a^5b^2d + 2441880a^5bc^2 + 4069800a^4b^3c + 813960a^3b^5 - 38760a^6g - 38760ab^6$
$- 232560a^5bf - 232560a^5ce - 116280a^5d^2 - 581400a^4b^2e - 581400a^2b^4c - 1162800a^4bcd - 193800a^4c^3$
$- 775200a^3b^3d - 1162800a^3b^2c^2 + 11628a^5h + 11628b^5c + 58140a^4bg + 58140a^4cf + 58140a^4de$
$+ 58140ab^4d + 116280a^3b^2f + 116280a^3c^2d + 116280a^3bd^2 + 116280ab^3c^2 + 116280a^2bc^3 + 116280a^2b^3e$
$+ 232560a^3bce + 348840a^2b^2cd - 3060a^4i - 3060b^4e - 3060ac^4 - 12240a^3bh - 12240a^3cg - 12240a^3df$
$- 12240ab^3f - 12240b^3cd - 18360a^2b^2g - 18360a^2c^2e - 18360a^2cd^2 - 18360ab^2d^2 - 6120a^3e^2 - 6120b^2c^3$
$- 36720a^2bcf - 36720a^2bde - 36720ab^2ce - 36720abc^2d + 680a^3j + 680b^3g + 680c^3d + 680ad^3$
$+ 2040a^2bi + 2040a^2ch + 2040a^2dg + 2040a^2ef + 2040ab^2h + 2040b^2cf + 2040b^2de + 2040ac^2f$
$+ 2040bc^2e + 2040bcd^2 + 2040abe^2 + 4080abcg + 4080abdf + 4080acde - 120a^2k - 120b^2i - 120c^2g$
$- 120d^2e - 120ce^2 - 120af^2 - 240abj - 240aci - 240adh - 240aeg - 240bch - 240bdg - 240bef$
$- 240cdf + 15al + 15bk + 15cj + 15di + 15eh + 15fg - m,$

$N = 2674440a^{14} - 17383860a^{12}b + 7726160a^{11}c + 42493880a^{10}b^2 - 3268760a^{10}d - 32687600a^9bc - 49031400a^8b^3$
$+ 1307504a^9e + 11767536a^8bd + 5883768a^8c^2 + 47070144a^7b^2c + 27457584a^6b^4 - 490314a^8f$
$- 3922512a^7be - 3922512a^7cd - 13728792a^6b^2d - 13728792a^6bc^2 - 27457584a^5b^3c - 6864396a^4b^5$
$+ 170544a^7g + 1193808a^6bf + 1193808a^6ce + 596904a^6d^2 + 596904a^2b^6 + 3581424a^5b^2e + 7162848a^5bcd$
$+ 1193808a^5c^3 + 5969040a^4b^3d + 5969040a^3b^4c + 8953560a^4b^2c^2 - 54264a^6h - 325584a^5bg - 325584a^5cf$
$- 325584a^5de - 325584ab^5c - 813960a^4b^2f - 813960a^4c^2d - 813960a^4bd^2 - 813960a^2b^4d - 1627920a^4bce$
$- 1085280a^3b^3e - 1085280a^3bc^3 - 3255840a^3b^2cd - 1627920a^2b^3c^2 - 7752b^7 + 15504a^5i + 15504b^5d$
$+ 77520a^4bh + 77520a^4cg + 77520a^4df + 77520ab^4e + 38760a^4e^2 + 38760b^4c^2 + 38760ba^2c^4$
$+ 155040a^3b^2g + 155040a^3c^2e + 155040a^3cd^2 + 155040a^2b^3f + 155040ab^2c^3 + 232560a^2b^2d^2$
$+ 310080a^3bcf + 310080a^3bde + 310080ab^3cd + 465120a^2b^2ce + 465120a^2bc^2d - 3876a^4j - 3876b^4f$
$- 3876bc^4 - 15504a^3bi - 15504a^3ch - 15504a^3dg - 15504a^3ef - 15504ab^3g - 15504b^3ce - 15504ac^3d$

$$
\begin{aligned}
&- 23256a^2b^2h - 23256a^2c^2f - 23256a^2be^2 - 23256b^2c^2d - 7752a^2d^3 - 7752b^3d^2 - 46512a^2bcg \\
&- 46512a^2bdf - 46512a^2cde - 46512ab^2de - 46512ab^2cf - 46512abc^2e - 46512abcd^2 + 816a^3k + 816b^3h \\
&+ 816c^3e + 816bd^3 + 2448a^2bj + 2448a^2ci + 2448a^2dh + 2448a^2eg + 2448ab^2i + 2448b^2cg + 2448b^2df \\
&+ 2448ac^2g + 2448bc^2f + 2448ad^2e + 2448ace^2 + 1224a^2f^2 + 1224b^2e^2 + 1224c^2d^2 + 4896abch \\
&+ 4896abdg + 4896abef + 4896acdf + 4896bcde - 136a^2l - 136b^2j - 136c^2h - 136d^2f - 136de^2 \\
&- 136bf^2 - 272abk - 272acj - 272adi - 272aeh - 272afg - 272bci - 272bdh - 272beg - 272cdg \\
&- 272cef + 16am + 16bl + 16ck + 16dj + 16ei + 16fh + 8g^2 - n, \\
P = &- 9694845a^{15} + 67863915a^{13}b - 30421755a^{12}c - 182530530a^{11}b^2 + 13037895a^{11}d + 143416845a^{10}bc \\
&+ 239028075a^9b^3 - 5311735a^{10}e - 53117350a^9bd - 26558675a^9c^2 - 239028075a^8b^2c - 159352050a^7b^4 \\
&+ 2042975a^9f + 18386775a^8be + 18386775a^8cd + 73547100a^7b^2d + 73547100a^7bc^2 + 171609900a^6b^3c \\
&+ 51482970a^5b^5 - 735471a^8g - 5883768a^7bf - 5883768a^7ce - 2941884a^7d^2 - 20593188a^6b^2e \\
&- 6864396a^6c^3 - 6864396a^3b^6 - 41186376a^6bcd - 41186376a^5b^3d - 61779564a^5b^2c^2 - 51482970a^4b^4c \\
&+ 245157a^7h + 245157ab^7 + 1716099a^6bg + 1716099a^6cf + 1716099a^6de + 5148297a^5b^2f + 5148297a^5c^2d \\
&+ 5148297a^5bd^2 + 5148297a^2b^5c + 10296594a^5bce + 8580495a^4b^3e + 8580495a^3b^4d + 8580495a^4bc^3 \\
&+ 25741485a^4b^2cd + 17160990a^3b^3c^2 - 74613a^6i - 74613b^6c - 447678a^5bh - 447678a^5cg - 447678a^5df \\
&- 447678ab^5d - 223839a^5e^2 - 1119195a^4b^2g - 1119195a^4c^2e - 1119195a^4cd^2 - 1119195a^2b^4e \\
&- 1119195ab^4c^2 - 2238390a^4bcf - 2238390a^4bde - 1492260a^3b^3f - 2238390a^3b^2d^2 - 2238390a^2b^2c^3 \\
&- 4476780a^3b^2ce - 4476780a^3bc^2d - 4476780a^2b^3cd - 873065a^3c^4 + 20349a^5j + 20349b^5e + 101745a^4bi \\
&+ 101745a^4ch + 101745a^4dg + 101745a^4ef + 101745ab^4f + 101745abc^4 + 101745b^4cd + 203490a^3b^2h \\
&+ 203490a^3c^2f + 203490a^3be^2 + 203490a^2b^3g + 203490a^2c^3d + 203490ab^3d^2 + 406980a^3bcg + 406980a^3bdf \\
&+ 406980a^3cde + 406980ab^3ce + 610470a^2b^2cf + 610470a^2b^2de + 610470a^2bc^2e + 610470a^2bcd^2 \\
&+ 610470ab^2c^2d + 67830a^3d^3 + 67830b^3c^3 - 4845a^4k - 4845b^4g - 19380a^3bj - 19380a^3ci - 19380a^3dh \\
&- 19380a^3eg - 19380ab^3h - 19380b^3cf - 19380b^3de - 19380ac^3e - 19380bc^3d - 19380abd^3 - 29070a^2b^2i \\
&- 29070a^2c^2g - 29070a^2d^2e - 29070a^2ce^2 - 29070ab^2e^2 - 29070b^2c^2e - 29070b^2cd^2 - 29070ac^2d^2 \\
&- 9690a^3f^2 - 58140a^2bch - 58140a^2bdg - 58140a^2bef - 58140a^2cdf - 58140ab^2cg - 58140ab^2df \\
&- 58140abc^2f - 116280abcde - 969c^5 + 969a^3l + 969b^3i + 969c^3f + 969cd^3 + 2907a^2bk + 2907a^2cj \\
&+ 2907a^2di + 2907a^2eh + 2907a^2fg + 2907ab^2j + 2907b^2ch + 2907b^2dg + 2907b^2ef + 2907ac^2h \\
&+ 2907bc^2g + 2907c^2de + 2907ad^2f + 2907bd^2e + 2907ade^2 + 2907bce^2 + 2907abf^2 + 5814abci \\
&+ 5814abdh + 5814abeg + 5814acdg + 5814acef + 5814bcdf - 153a^2m - 153b^2k - 153c^2i - 153d^2g \\
&- 153cf^2 - 153ag^2 - 306abl - 306ack - 306adj - 306aei - 306afh - 306bcj - 306bdi - 306beh \\
&- 306bfg - 306cdh - 306ceg - 306def - 51e^3 + 17an + 17bm + 17cl + 17dk + 17ej + 17fi + 17gh - p.
\end{aligned}
$$

The values of the coefficients to I were found by the solution of the equations obtained from (3). I then discovered a law by means of which the remaining coefficients were determined. In each coefficient the sum of the numerical coefficients is -1 or $+1$, according as the number of the coefficient is odd or even. This may be applied as a test of the accuracy of the results.

Mr. *Seitz* also computed the value of Q, which we reluctantly omit for want of space.

Prof. *Trowbridge* employed the Differential Calculus in his solution, and found the values of the first seven letters.

LIST OF CONTRIBUTORS TO THE JUNIOR DEPARTMENT.

The following persons have furnished solutions to the Problems indicated by the numbers:

E. B. Seitz, Greenville, O., 35, 36, 38, 40, 41, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 54 and 55; K. S. Putnam, Rome, N. Y., 35, 36, 37, 38, 39, 40, 41, 43, 44, 45, 46, 47, 48, 49, 50, 51 and 53; B. F. Burleson, Oneida Castle, N. Y., 35, 36, 37, 38, 40, 41, 43, 44, 45, 46, 47, 48, 49, 52 and 53; Prof. F. P. Matz, King's Mountain, N. C., 35, 37, 38, 39, 40, 41, 46, 47, 48, 49, 52 and 54; Prof. Frank Albert, Millersville, Pa., 35, 36, 38, 40, 43, 44, 46, 47, 48, 49, 50 and 51; John I. Clark, Moran, Ind., 35, 36, 37, 38, 40, 42, 43, 45, 47, 48 and 49; Marcus Baker, Washington, D. C., 40, 41, 44, 45, 47, 48, 49 and 52; William Wiley, Detroit, Mich., 35, 38, 40, 44, 45, 47, 48 and 49; Sylvester Robins, North Branch Depot, N. J., 39, 40, 41, 47, 49 and 51; O. D. Oathout, Read, Iowa, 35, 40, 45, 47, 48 and 49; V. Webster Heath, Rodman, N. Y., 35, 37, 40, 42, 47 and 48; D. W. K. Martin, Webster, O., 35, 36, 38, 40 and 45; Prof. D. J. McAdam, Washington, Pa., 44, 45, 46, 47 and 53; Gavin Shaw, Kemble, Ontario, Canada, 35, 38, 40 and 44; William Hoover, Bellefontaine, O., 44, 50, 51 and 54; George H. Leland, Windsor, Vt., 37, 40, 41 and 42; Walter S. Nichols, New York, N. Y., 35, 36, 30 and 49; E. P. Norton, Allen, Mich., 35, 40, 41 and 47; J. R. Fagan, Erie, Pa., 35, 36, 38 and 39; Theo. L. DeLand, Washington, D. C., 35, 40 and 47; E. J. Edmunds, New Orleans, La., 40, 44 and 47; Henry Heaton, Sabula, Iowa, 37, 39 and 40; W. E Heal, Wheeling, Ind., 40, 46 and 47; G. G. Washburn, North East, Pa., 35, 36 and 38; Mrs. Anna T. Snyder, Chicago, Ill., 45 and 47; Prof. David Trowbridge, Waterburg, N. Y., 47 and 55; Walter Siverly, Oil City, Pa., 43 and 44; O. H. Merrill, Mannsville, N. Y., 35 and 47; Benjamin Headley, Dillsborough, Ind., 37 and 42; Dr. David S. Hart, Stonington, Conn., 51; Prof. H. T. J. Ludwig, Mt. Pleasant, N. C., 49; Prof. J. F. W. Scheffer, Mercersburg, Pa., 46; F. F. Leary, Covington, Ky., 40; T. P. Stowell, Rochester, N. Y., 37; Milton Rice, Moran, Ind., 42; J. B. Sanders, Bloomington, Ind., 36; Prof. Joseph Ficklin, Columbia, Mo., 48; Prof. Edward Brooks, Millersville, Pa., 38.

The first prize is awarded to E. B. Seitz, Greenville, O., and the second prize to Prof. David Trowbridge, Waterburg, N. Y.

PROBLEMS.

88.—Proposed by JOHN I. CLARK, Moran, Clinton County, Indiana.

When silver is 4 per cent. discount and gold is at 5 per cent. premium, taking greenbacks as a standard, what will silver be worth if we take gold as a standard.

89.—Proposed by G. G. WASHBURN, North East, Erie County, Pennsylvania.

Two men. A and B, each desire to sell a horse to C; A asks a certain price and B asks 50 per cent. more. C refused to pay either price; A then reduced his price $16\frac{2}{3}$ per cent. and B reduced his $33\frac{1}{3}$ per cent., at which prices C took both horses, paying $220 for them. What was each man's asking price?

90.—Proposed by K. S. PUTNAM, Rome, Oneida County, New York.

Having cut a square, area a^2, from one corner of a board $2a$ wide and $8a$ long, cut the remainder into three pieces so that they will make a square.

91.—Proposed by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

Show that in any isosceles triangle, the square of a line drawn from the vertex to any point in the base, plus the product of the segments of the base, is constant.

92.—Proposed by JAMES Q. BRIGHAM, Walton, Harvey County, Kansas.

What is the rate per cent. of interest when a sum of money amounts to ten times itself in 21 years, compounded annually? And what would be the rate for the same time if compounded semi-annually?

93.—Proposed by E. J. EDMUNDS, B. S., New Orleans, Orleans County, Louisiana.

A point P being given on the base of a triangle, draw a line across the triangle parallel to the base which will subtend a right angle at P.

94.—Proposed by HENRY NICHOLS, Hampton, Rock Island Co., Ill.; and Mrs. ANNA T. SNYDER, Chicago, Cook Co., Ill.

It is required to divide a tapering board into two equal parts by sawing it across, parallel to the ends. Find the width at each end so that the lengths of the pieces will be expressed by rational numbers.

95.—Proposed by T. A. KINNEY, St. Albans, Franklin County, Vermont.

Given two circles and a point without them to draw through the point a line cutting the two circles so that the portion of the line intercepted between the two circles shall be equal to a given line.—[From *Chauvenet's Geometry*.

96.—Proposed by L. C. WALKER, New Madison, Darke County, Ohio.

A cannon ball, radius r, rolls into the corner of a room whose walls are at right angles, and perpendicular to the floor. What is the radius of another ball just touched by the cannon ball?

97.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Solve, by quadratics, the equation $x^4 - 2ax^3 - 2abx + b^2 = 0$.

98.—Proposed by Mrs. ANNA T. SNYDER, Chicago, Cook County, Illinois.

On a circular griddle 12 inches in diameter, 3 equal circular cakes are baked of such size that each cake touches the edge of the griddle and the edges of the other two cakes. What is the diameter of each cake?

99.—Proposed by W. WOOLSEY JOHNSON, Professor of Mathematics, St. Johns College, Annapolis, Anne Arundel Co., Md.

The extremities of a line of fixed length which slides along a fixed line are joined to two fixed points. Find the locus of the intersection of the joining lines.

100.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Monroe County, Indiana.

Required to find a number such that when it is added to 15, 27 and 45 there arise three numbers which are in geometrical progression.

101.—Proposed by O. H. MERRILL, Mannsville, Jefferson County, New York.

Two equal circles intersect, the center of each being on the circumference of the other. From A, one of the points of intersection, a diameter AB of one of the circles is drawn. Find the radius of the circle touching AB and the circumferences of both the given circles.

102.—Proposed by J. D. WILLIAMS, Superintendent of Public Schools, Sturgis, St. Joseph County, Michigan.

Solve the equations $x + y + xy = 75$, $x^2 - y^2 = 315$, by quadratics.

103.—Proposed by WILLIAM WOOLSEY JOHNSON, Professor of Mathematics, St. John's College, Annapolis, Anne Arundel Co., Md.

BR is an ordinate, from any point B of a circle, to the diameter passing through the fixed point A; and T is the intersection of the tangent at A with the radius produced through B. Find the locus of the intersection of AB and TR.

104.—Proposed by W. T. R. BELL, M. A., Principal King's Mountain High School, King's Mountain, Cleaveland Co., N. C.

The railroad debt of a certain company is $56400, interest at 7 per cent., coupons payable semi-annually. What tax must be levied to pay the interest and create such a sinking fund as will absorb the debt in 15 equal annual payments?

105.—Proposed by JOHN REA, Hill's Fork, Adams County, Ohio.

Find the co-ordinates a, b, a', b', of the centers, and the radii r, r' of the two circles

$$y^2 + x^2 - 20x - 40 = 0, \text{ and } x^2 + y^2 + 40y + 50 = 0;$$

and in case the two circumferences cut each other, what are the co-ordinates of the points of intersection?—[From *Loomis' Analytical Geometry*.

106.—Proposed by S. C. Brace, Philadelphia, Pennsylvania.

A circular saw, one foot in diameter, in cutting off a round log, is stopped when it reaches the center of the log, when it is found that the wood covers $\frac{1}{3}$ of the side-surface of the saw. What is the diameter of the log?

107.—Proposed by Sylvester Robins, North Branch Depot, Somerset County, New Jersey.

At a Firemen's Fair a silver trumpet is offered to the company exhibiting the ladder that can be used in the greatest number of streets and alleys for the purpose of reaching windows on either side without changing the location of its foot. All bases and perpendiculars must be rational lengths, and a company may include in their count dimensions having as many decimal places as their ladder has, but no more. The "Hudsons" bring a ladder 65 feet long, the "Keystones" offer one $32\frac{1}{2}$ feet in length, and the "Delawares" show one of $42\frac{1}{4}$ feet. To whom must the trumpet be awarded, and on what count?

108.—Proposed by Justus F. W. Scheffer, Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin County, Pennsylvania.

Three army corps, A, B and C, being engaged in a campaign, have provisions sufficient for 30 weeks. On these provisions B and C would subsist 9 weeks longer than A and B; and A and C, 15 weeks longer than B and C. After 6 weeks the three army corps encounter the enemy, in consequence of which A loses $\frac{1}{8}$ of its troops, B loses $\frac{1}{4}$ and C loses $\frac{1}{5}$; also $\frac{1}{8}$ of the remaining provisions are lost. How many weeks will the remainder of the three corps subsist on the provisions left?

109.—Proposed by Thomas Bagot, Principal Shelby School, Canaan, Jefferson County, Indiana.

A field in the form of a right-angled triangle has a base and perpendicular of 40 and 200 feet respectively. What length of rope attached at the vertex of the right angle will permit a horse to graze upon half the field?

110.—Prize Problem. Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Give an expeditious method of approximating to the square root of a quantity, and find by it the square root of 2 to at least one hundred and fifty places of decimals.

Solutions of these problems should be received by September 1, 1879.

SENIOR DEPARTMENT.

Solutions of Problems proposed in No. 2.

56.—Proposed by George Eastwood, Saxonville, Middlesex County, Massachusetts.

In a plane, the equation of a straight line in terms of the perpendicular (p) from the origin and the angle (θ) which it makes with the axis being $y\sin\theta + x\cos\theta = p$; prove that the same form holds for the equation of a great circle of the sphere, when x, y, p are put for $\tan x$, $\tan y$ and $\tan p$.

Solution by W. E. Heal, Wheeling, Delaware County, Indiana.

We will assume that the required equation is reduced to the form $A\tan x + B\tan y = \tan p$; we wish to find the coefficients A and B. Making $y = 0$ we find for the intercept on the axis of x,

$$A\tan x_1 = \tan p, \quad \therefore A = \frac{\tan p}{\tan x_1}. \quad \text{Similarly, } B\tan y_1 = \tan p, \text{ and } B = \frac{\tan p}{\tan y_1}.$$

But by Spherical Trigonometry we have $\frac{\tan p}{\tan x_1} = \cos\theta$, $\frac{\tan p}{\tan y_1} = \sin\theta$; $\therefore A = \cos\theta$, $B = \sin\theta$ and the equation is reduced to $\cos\theta\tan x + \sin\theta\tan y = \tan p$. Q. E. D.

Solved also in an elegant manner by *E. B. Seitz*, *F. P. Matz* and the *Proposer*.

57.—Proposed by Oscar H. Merrill, Mannsville, Jefferson County, New York.

Prove that the cube of any given number is greater than the product of any other three numbers whose sum is three times the given number.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Let m be the given number, $m + 2n$ one of the other numbers. Then the product of the three numbers will be greatest when the remaining two numbers are equal; hence let $m - n$ be each of the two numbers. Then the product of the three numbers $= (m + 2n)(m - n)^2 = m^3 - 3mn^2 + 2n^3$, which is less than m^3 for all admissible values of n.

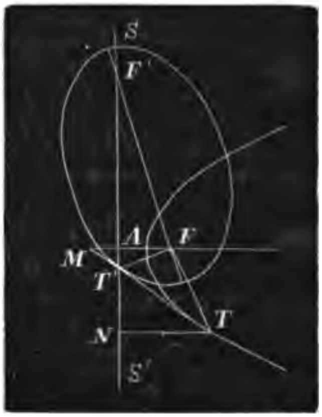
Good solutions given by Messrs. *Heal*, *Nichols*, *Trowbridge* and the *Proposer*.

58.—Proposed by Miss Christine Ladd, B. A., Baltimore, Maryland.

An ellipse and a parabola have a common focus and the other focus of the ellipse moves on the directrix of the parabola. Show that the points of contact of a common tangent subtend a right angle at the common focus.

Solution by E. B. SEITZ, Greenville, Darke County, Ohio; and DUNLAP J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania.

Let F be the common focus, F′ the other focus of the ellipse, SS′ the directrix of the parabola, T′ the intersection of SS′ and the ellipse, TT′ the tangent to the parabola, M the point in which TT′ produced meets the axis of the parabola produced, and TN the perpendicular on SS′.



Now $\angle$ FTM $= \angle$ FMT $= \angle$ NTT′, and TF = TN; $\therefore \angle$ TT′F $= \angle$ TT′N, and $\angle$ TFT′ $= \angle$ TNT′ = a right angle.

Again, $\angle$ F′T′M $= \angle$ TT′N $= \angle$ FT′T; $\therefore$ TT′ is a tangent to the ellipse.

Solved also by the *Proposer.*

59.—Proposed by Dr. S. H. WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, Yates County, N. Y.

When the sun's declination was δ north he rose β° farther south than when his declination was δ' north. Required my latitude.

Solution by the PROPOSER.

Let P be the north pole, and P′ the north point of the horizon, S the place of the sun when its declination is δ, and S′ when δ'.

Then SS′ $= \beta$, SP = co-declination of δ, and S′P = co-declination of δ' and the required latitude = PP′ $= \lambda$. In the spherical triangle SPS′ all the sides are given to find the angle PSS′ $= A$. We have, therefore $A = 2\cos^{-1}\sqrt{\left(\frac{\cos\frac{1}{2}(\delta+\delta'-\beta)\sin\frac{1}{2}(\delta'+\beta-\delta)}{\sin\beta\cos\delta}\right)}$ and $\lambda = \sin^{-1}(\sin A \sin\delta)$. If $\delta = 10^\circ$, $\delta' = 20^\circ$, and $\beta = 12^\circ$, then $\lambda = 32^\circ\ 12'\ 41''$.

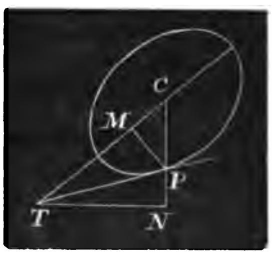
Solved also by *E. B. Seitz* and Prof. *De Volson Wood.*

60.—Proposed by GEORGE LILLEY, Kewanee, Henry County, Illinois.

Find the minimum eccentricity of an ellipse capable of resting in equilibrium on a perfectly rough inclined plane, inclination β.

I.—Solution by E. B. SEITZ, Greenville, Darke County, Ohio.

Let TP represent the inclined plane, P the point at which the ellipse touches the plane, C the center of the ellipse, T the point at which the major axis produced meets the plane, TN a horizontal line, and let CP be vertical. Let a, b = the semi-axes of the ellipse, CM $= x$, PM $= y$, angle CTP $= \theta$, and angle PTN $= \beta$. Then we have $\tan\theta = \frac{b^2x}{a^2y}$ ...(1), and $\tan(\theta+\beta) = \frac{x}{y}$ ...(2).



From (1) and (2) we find $a^2\tan\theta = b^2\tan(\theta+\beta)$..(3). By development and reduction (3) becomes $a^2\tan\beta\tan^2\theta - (a^2-b^2)\tan\theta + b^2\tan\beta = 0$.....(4) The two positions of an ellipse, in which its center is vertically above the point of contact with the plane, are given by the values of $\tan\theta$ in (4). When the ellipse is in equilibrium, the value of $\tan\theta$ is between the two values in (4). Hence, for the ellipse of minimum eccentricity these two values are equal, and therefore from (4) we must have $(a^2-b^2)^2 = 4a^2b^2\tan^2\beta$...(5). Solving (5) for b^2, we find $b^2 = a^2 - \frac{2a^2\sin\beta}{1+\sin\beta}$...(6). Substituting $a^2(1-e^2)$ for b^2, we find $e^2 = \frac{2\sin\beta}{1+\sin\beta}$.

II.—Solution by DE VOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

The center of the ellipse will be vertically over the point of support, and since the plane is tangent to the ellipse, a vertical through the point of support and a parallel to the plane through the center will make conjugate diameters. Hence the acute angle of the conjugate diameters is $90^\circ - \beta$.

At the point bordering on motion the potential energy is a maximum, and the major axis will bisect the acute angle of the conjugate diameters; hence the positive angles made by the conjugate diameters with the major axis will be $\theta = 45^\circ - \frac{1}{2}\beta$, $\theta' = 135^\circ + \frac{1}{2}\beta$.

The condition for conjugate diameters is $a^2\sin\theta\sin\theta' + b^2\cos\theta\cos\theta' = 0$, which, by substituting the preceding values, gives $a^2\sin^2(45^\circ - \frac{1}{2}\beta) - b^2\cos^2(45^\circ - \frac{1}{2}\beta) = 0$; which reduced gives

$$\sqrt{\left(\frac{a^2-b^2}{a^2}\right)} = e = \sqrt{\left(\frac{2\sin\beta}{1+\sin\beta}\right)}.$$

Solved also by Prof. *F. P. Matz.*

61.—Proposed by Miss Christine Ladd, B. A., Baltimore, Maryland.

If ABC be a triangle inscribed in a conic, $P'P''P'''$ the points in which its sides meet the directrix of the conic, and $Q'Q''Q'''$ the poles of focal chords through $P'P''P'''$, then will AQ', BQ'', CQ''' meet in a point.

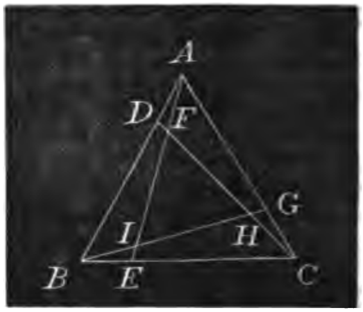
Solution by the Proposer.

Reciprocating, this becomes: If from the vertices of a triangle ABC, lines be drawn to I, the center of the inscribed circle, and perpendiculars to them, IM', IM'', IM''', meet opposite sides in M', M'', M''', then are M', M'', M''' three collinear points. And this is true, since the equations to the lines in question are $\sin\frac{1}{2}B\cos\frac{1}{2}A\alpha + \sin\frac{1}{2}A\cos\frac{1}{2}B\beta - \cos\frac{1}{2}C\gamma = 0$, &c., and the lines joining to vertices their intersections with opposite sides are $\alpha\cos\frac{1}{2}A + \beta\cos\frac{1}{2}B = 0$, &c., therefore, &c.

62.—Proposed by F. P. Matz, M. E., M. S., Professor of Higher Mathematics and Astronomy, King's Mountain High School, King's Mountain, Cleaveland County, North Carolina.

In an equilateral triangle ABC lines are drawn as in Problem 8; find the average area of the equilateral triangle formed by the intersections of the lines.

Solution by the Proposer; Prof. DeVolson Wood, M. A., C. E.; William Hoover; Walter S. Nichols; and E. B. Seitz.



Put $AB = a$ and $AD = x$, then $CD = \sqrt{(a^2 - ax + x^2)}$,

$$CH = \frac{ax}{\sqrt{(a^2 - ax + x^2)}},\ DF = \frac{x^2}{\sqrt{(a^2 - ax + x^2)}},\ FH = \frac{a^2 - 2ax}{\sqrt{(a^2 - ax + x^2)}},$$

and the required average area is

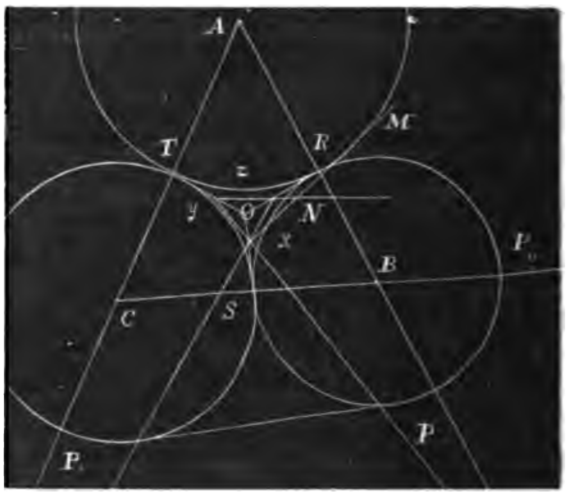
$$\varDelta = \frac{\frac{\sqrt{3}}{4}a^2\int_0^{\frac{1}{2}a}\frac{(a-2x)^2dx}{a^2 - ax + x^2}}{\int_0^{\frac{1}{2}a}dx}, = \frac{a\sqrt{3}}{2}\int_0^{\frac{1}{2}a}\frac{(a-2x)^2dx}{a^2 - ax + x^2},$$

$$= \frac{a\sqrt{3}}{2}\left[4x - 2a\tan^{-1}\left(\frac{a-2x}{a\sqrt{3}}\right)\right]_0^{\frac{1}{2}a}, = a^2(\sqrt{3} - \tfrac{1}{2}\pi).$$

63.—Proposed by Isaac H. Turrell, Cumminsville, Hamilton County, Ohio.

Within the space enclosed by three given circles which touch each other externally it is required to inscribe, geometrically, three circles each of which shall touch the other two and also two of the given circles. Analytical solutions also desired.

I.—Solution by the Proposer.



Find the external centers of similitude $PP_{,,}P_{,}$ and draw the radical axes OR, OS, OT, of the pairs of given circles AB, BC, CA, and designate by X Y, Z, (which are not shown in the diagram,) the circles required to lie respectively opposite A, B and C.

If a pair of circles be touched in the same way by another pair, the center of similitude of either pair lies on the radical axis of the other pair; hence x, y and z, the external centers of similitude of the pairs AX, BY, CZ, lie respectively on OS, OT, OR; and since AX is touched in the same way by the pairs BY, CZ, BC, its radical axis contains the points y, z, $P_{,,}$. Similarly y, x, P, and z, x, $P_{,}$ are each three points *in directum*. Hence xyz may be regarded as a variable triangle whose vertices move along the fixed lines OS, OT, OR, which meet in a point, and whose sides pass respectively through $PP_{,,}P_{,}$ three given points in a right line: in other words, if the sides xy, yz, for all positions of the variable triangle, pass through the points P, $P_{,,}$, the third side zx will always pass through $P_{,}$. (*Mulcahy's Modern Geometry*, p. 18.)

Since the radical axis of a pair of circles is perpendicular to the line joining their centers, the problem is reduced to finding that position of the variable triangle in which the lines Ax, yz, shall intersect at right angles in some point V. On $AP_{,,}$ as a diameter, describe a circle on which the point V must lie. From A draw any three lines meeting the circle $AP_{,,}$ in $V'V''V'''$, and the line OS in $x_{,}x_{,,}x_{,,,}$.

Draw $Px_{,}$, $Px_{,,}$, $Px_{,,,}$ meeting OR in $z_{,}$, $z_{,,}$, $z_{,,,}$; also $P_{,}z_{,}$, $P_{,}z_{,,}$, $P_{,}z_{,,,}$ meeting the circle $AP_{,,}$ in $V_{,}V_{,,}V_{,,,}$. (To avoid complication these lines, as well as the circle $AP_{,,}$ are omitted in the figure; they can, however, be easily reproduced.)

The anharmonic ratio A . VV′V″V‴ = $P_{,}$. $xx_{,}x_{,,}x_{,,,}$ = $P_{,,}$. $zz_{,}z_{,,}z_{,,,}$ = $P_{,,}$. $VV_{,}V_{,,}V_{,,,}$; hence we have, on the circle $AP_{,,}$, six given points, V′V″V‴$V_{,}V_{,,}V_{,,,}$ to find a seventh V, such that the anharmonic ratio of $VV_{,}V_{,,}V_{,,,}$ shall be equal to that of VV′V″V‴.

This is an elementary problem, (*Mulcahy*, p. 22,) and gives two positions for the required point V, which shows that the original problem admits of *two* solutions; one position gives the group *enclosed by* the circles ABC, the other the group *enclosing them.*

To complete the solution, draw the line AV meeting OS in x, $P_{,}x$ meeting OR in z, and $P_{,,}z$ meeting OT in y. To determine the circle X, draw a tangent xM to the circle A, intersecting $P_{,,}z$ in N; from N lay off towards x a distance Nq = NM. Since $P_{,,}z$ is the radical axis of the circles A and X, the point q (not shown in the figure) will be the point of tangency on xM of the circle X whose center will evidently be the intersection of Ax with a line through q perpendicular to xM. Similarly the circles Y, Z can be determined.

NOTE.—A similar construction will apply if A, B and C are three small circles drawn on the surface of a sphere.

II.—Solution by E. B. Seitz.

Let A, B, C be the centers of the three given circles, radii a, b, c; A′, B′, C′ the centers of the three inscribed circles, radii x, y, z, the circle A′ touching B and C, B′ touching A and C, and C′ touching A and B. Put $AB = a + b$, $AC = a + c$, $BC = b + c$, $A'b = b + x$, $A'C = c + x$, $AA' = s$, $\angle ABC = \beta$, $\angle ABA' = \theta$, $\angle CBA' = \varphi$, and let t = the common tangent of the circles A and A′.

Now, since each of the six circles A, B, C, A′, B′, C′ touches four of the others externally, the square of the common tangent of any two of the circles, which do not touch each other, is equal to twice the product of their diameters (*Analyst*, vol. ii, p. 24);

$$\therefore \quad t^2 = 8ax, \text{ and } s^2 = (a-x)^2 + t^2 = a^2 + 6ax + x^2 \qquad (1).$$

By Trigonometry we have

$$\cos\theta = \frac{(a+b)^2 + (b+x)^2 - s^2}{2(a+b)(b+x)} = 1 - \frac{4ax}{(a+b)(b+x)}; \;\therefore\; \sin\tfrac{1}{2}\theta = \sqrt{\left(\frac{2ax}{(a+b)(b+x)}\right)}.$$

Similarly we find

$$\sin\tfrac{1}{2}\varphi = \sqrt{\left(\frac{cx}{(b+c)(b+x)}\right)}, \; \sin\tfrac{1}{2}\beta = \sqrt{\left(\frac{ac}{(a+b)(b+c)}\right)}, \text{ and } \cos\tfrac{1}{2}\beta = \sqrt{\left(\frac{b(a+b+c)}{(a+b)(b+c)}\right)}.$$

But $\sin\frac{1}{2}(\theta+\varphi) = \sin\frac{1}{2}\beta$, whence $\sin^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\varphi - \sin^2\frac{1}{2}\beta + 2\cos\frac{1}{2}\beta\sin\frac{1}{2}\theta\sin\frac{1}{2}\varphi = 0$ ....(2).

Substituting the values of $\sin\frac{1}{2}\theta$, &c., in (2), clearing and reducing, we get an equation of the first degree, from which we readily find $x = \dfrac{abc}{2ab + 2ac + bc + 2\sqrt{[2abc(a+b+c)]}}$.

By symmetry we have $y = \dfrac{abc}{2ab + ac + 2bc + 2\sqrt{[2abc(a+b+c)]}}$, $z = \dfrac{abc}{ab + 2ac + 2bc + 2\sqrt{[2abc(a+b+c)]}}$.

64.—Proposed by J. J. Sylvester, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematic Johns Hopkins University, Baltimore, Maryland.

If there be two equations in x, (which for greater simplicity may be supposed to be of the same degree n), find the most general form of M = a rational integral function of the coefficients of these equations such that $Mx, Mx^2, \ldots Mx^{n-1}$ shall each of them also be rational functions of the same.

Solution by Rev. W. J. Wright, M. A., Ph. D., Member of the London Mathematical Society, Cape May Point, Cape May Co., N. J.

It is assumed that the equations are simultaneous, then M is the eliminant of the given equations, and it follows as a corollary that Mx, &c., are rational integral functions of the coefficients.

Let $x^n + a_1x^{n-1} + a_2x^{n-2} + \text{\&c.} = 0$..................(1), and $x^n + b_1x^{n-1} + b_2x^{n-2} + \text{\&c.} = 0$..................(2) be a pair of simultaneous equations, α, β, γ, &c., the roots of (1), $\alpha, \beta_1, \gamma_1$, &c., those of (2), in which α is the common root. If the roots of (1) be substituted in (2) it is plain that (2) will vanish for the value α and therefore the entire product of all these successive substitutions will vanish. But by the Theory of Equations this product is a symmetric function of the roots of (1) and is expressible in terms of its coefficients.

Now $\Sigma\alpha = -a_1$, $\Sigma\alpha^2 = a_1^2 - 2a_2$, $\Sigma\alpha\beta = a_2$, &c.; consequently by introducing the values of these symmetric functions we evidently obtain a homogeneous function of the coefficients of the two equations which is the eliminant. Mx, Mx^2, &c., must also be functions of these coefficients since in the course of elimination by Bezout's or any other method we must arrive at an equation of the first degree from which the value of x is found in terms of the coefficients of one or both of the equations, and this multiplied by M gives us still a rational function of the coefficients.

Illustration.—$Mx^{n-1} = M = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0$ is the eliminant of $ax + b = 0$, $a_1x + b_1 = 0$. $Mx = \begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \times -\dfrac{b}{a}$.

See my *Tract on Invariants*, Arts. 1—7.

65.—Proposed by Dr. David S. Hart, M. A., Stonington, New London County, Connecticut.

Find six square numbers whose sum is a square, and the sum of their roots plus the root of their sum a square.

Solution by the Proposer.

Let $2ms$, $2ns$, $2ps$, $2qs$, $2rs$ and $m^2+n^2+p^2+q^2+r^2-s^2$ be the roots of the squares; then the sum of their squares is $(m^2+n^2+p^2+q^2+r^2+s^2)^2$, whose root is $m^2+n^2+p^2+q^2+r^2+s^2$, which added to the sum of their roots gives $(2m+2n+2p+2q+2r)s+2(m^2+n^2+p^2+q^2+r^2)=\square=4t^2$;

whence $s=\dfrac{2t^2-(m^2+n^2+p^2+q^2+r^2)}{m+n+p+q+r}$, in which expression m, n, p, q, r, may be consecutive numbers in the natural series, and t such that s may be nearly equal to one of these numbers.

Let $m=2$, $n=3$, $p=4$, $q=5$, $r=6$ and $t=10$; then $s=\frac{11}{2}$, whence the roots of the squares are 22, 33, 44, 55, 66 and $\frac{239}{4}$, and multiplying by 4, we have 88, 132, 176, 220, 264 and 239, the sum of whose squares is $(481)^2$, and their sum is 1119, $\therefore$ $1119+481=1600=(40)^2$, as required.

From the solution above we readily see how any n numbers may be found having the same conditions.

66.—Proposed by Benjamin Peirce, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the United States Coast Survey, Cambridge, Middlesex County, Massachusetts.

What are the probabilities at a game of a given number of points, but at which there is only one person who is the actual player? When the player is successful he counts a point, but when he is unsuccessful he loses all the points which he has made and adds one point to the score of his opponent.

Solution by the Proposer.

Let $\varphi(i,j)$ denote the probability that the player will gain the game, when he has i points to make, and when the opponent has j points to make. Let h be the whole number of points of the game, and a the player's chance of gaining a point. It is obvious that $\varphi(i,j)=a\varphi(i-1,j)+(1-a)\varphi(h,j-1)$, which, when $j=1$, gives $\varphi(i,1)=a\,\varphi(i-1,1)=a^i$; and in general $\varphi(i,j)=a^i+(1-a^i)\,\varphi(h,j-1)$, because $\varphi(i,j)-\varphi(h,j-1)=a[\varphi(i-1,j)-\varphi(h,j-1)]$.

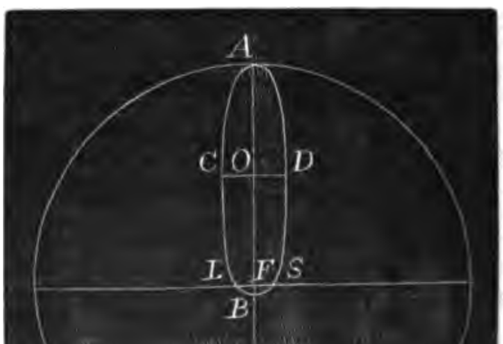
This gives $\varphi(h,j)-1=(1-a^h)[\varphi(h,j-1)-1]$, whence $\varphi(h,j)=1-(1-a^h)^j$,

$\varphi(i,j)=1-(1-a^i)(1-a^h)^{j-1}$, and at the beginning of the game $\varphi(h,h)=1-(1-a^h)^h$.

67.—Proposed by Dr. Joel E. Hendricks, M. A., Editor of the *Analyst*, DesMoines, Polk County, Iowa.

Suppose the earth to be an indefinitely thin spherical shell, equatorial radius, rotary velocity and gravitating force the same as at present except that gravity is assumed to be all concentrated at the center, and none in the shell; and suppose a particle at the equator, and on the inside of the shell, to be separated from the shell, and to move henceforth from its centrifugal force, resulting from its motion while attached to the shell, and from the gravitating force at the earth's center. Required the axes of the ellipse the particle will describe, and the time required for its return to the same point in space at which it was detached from the shell.

Solution by the Proposer.



Let r denote the equatorial radius of the earth $=3962\frac{1}{2}$ miles, say; let v = equatorial velocity per second $=\dfrac{\pi r}{p}$, where $2p$ = interval, in seconds, of one rotation of the earth $=86164$ seconds. Then is $v^2=\dfrac{\pi^2r^2}{p^2}$ ...(1). Let v' = the velocity per second of a body, supposed to move in a circular orbit about the center of the earth, under the influence of a centripetal force g, and at the distance r from the earth's center. Then, from the known laws of orbital motion, we readily find $v'=\sqrt{(gr)}$;

$\therefore\quad v'^2=gr$.......................(2).

Let a, b represent the semi-axes of the required ellipse AB, F, the center of the earth, and consequently one focus of the ellipse, and let $l=$ LS $=$ the *latus rectum* of the ellipse. Then, by Cor's 1 and 2, Prop. XVI, Book I, Principia, $l=\dfrac{2rv^2}{v'^2}=\dfrac{2b^2}{a}$ (by property of the ellipse)...(3). Substituting for v^2 and v'^2 in (3) their values from (1) and (2), $l=\dfrac{2\pi^2r^2}{gp^2}=\dfrac{2b^2}{a}=27.374$ miles.......(4). We have FO $=r-a=\sqrt{(a^2-b^2)}$, by property of the ellipse; whence $b^2=2ar-r^2$......(5). Substituting this value of b^2 in (4) and reducing, we get $a=\dfrac{grp^2}{2gp^2-\pi^2r}\doteq1984.677$ miles..(6).

And this value of a in (5) gives $b^2=\dfrac{\pi^2r^3}{2gp^2-\pi^2r}$, or $b=\pi r\sqrt{\left(\dfrac{r}{2gp^2-\pi^2r}\right)}=164.817$ miles.

Having determined the values of a and b, we have, by Kepler's second law,

$$\tfrac{1}{2}\rho v_1 dt : dt :: \pi ab : t, \text{ or } \tfrac{1}{2}\rho v_1 : 1 :: \pi ab : t,$$

ρ being the radius-rector and v_1 the velocity in the ellipse. But for equal portions of time ρv_1 is constant by Kepler's second law, therefore, if the unit of time be one second, $\rho v_1 = rv$, and consequently $t = \frac{2abp}{r^2} = 29'\ 55''$. Therefore the particle will return to the point in space at which it was detached, in 29 minutes and 55 seconds.

Solved in a similar manner by *E. B. Seitz.*

68.—Proposed by Prof. DAVID TROWBRIDGE, M. A., Waterburg, Tompkins County, New York.

A sphere is divided at random by a plane, and then two points are taken at random within the sphere; find the chance that both points are on the same side of the plane.

I.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Let x be the distance of the dividing plane from the center of the sphere; then the sphere is divided into two segments having a common base, radius $\sqrt{(r^2 - x^2)}$, and altitudes $r - x$ and $r + x$.

Let V = volume of the smaller segment; then

$V = \pi(r^2 - x^2) \times \frac{1}{2}(r - x) + \frac{1}{6}\pi(r - x)^3, = \frac{1}{3}\pi(2r^3 - 3r^2x + x^3)$, and $\frac{4}{3}\pi r^3 - V$ = volume of the other segment. The chance that both points are in the segment V is $\frac{V^2}{(\frac{4}{3}\pi r^3)^2}$, and that both are in the other segment, $\frac{(\frac{4}{3}\pi r^3 - V)^2}{(\frac{4}{3}\pi r^3)^2}$, for a particular value of x.

Hence the chance required is

$$p = \int_0^r \left(\frac{V^2 + (\frac{4}{3}\pi r^3 - V)^2}{(\frac{4}{3}\pi r^3)^2}\right) dx \div \int_0^r dx, = 1 - \frac{3}{2\pi r^4}\int_0^r V dx + \frac{9}{8\pi^2 r^7}\int_0^r V^2 dx = \frac{26}{35}.$$

Solved in a similar manner by *Walter S. Nichols.*

II.—Solution by E. B. SEITZ, Greenville, Darke County, Ohio.

Let x = the altitude of one of the segments into which the sphere is divided; then the volume of the segment is $\frac{1}{3}\pi x^2(3r - x)$, and the required chance is

$$p = \frac{\int_0^{2r} 2\left[\frac{1}{3}\pi x^2(3r - x)\right]^2 dx}{\int_0^{2r} (\frac{4}{3}\pi r^3)^2 dx}, = \frac{1}{16 r^7}\int_0^{2r} (3r - x)^2 x^4 dx, = \frac{26}{35}.$$

If n points be taken at random within the sphere, the chance that they will all be on the same side of the plane is

$$p = \frac{\int_0^{2r} 2\left[\frac{1}{3}\pi x^2(3r - x)\right]^n dx}{\int_0^{2r} (\frac{4}{3}\pi r^3)^n dx}, = \frac{1}{2^{2n} r^{3n+1}}\int_0^{2r} (3r - x)^n x^{2n} dx,$$

$$= \frac{1}{2^{2n} r^{3n+1}}\left[\frac{x^{2n+1}(3r - x)^n}{2n + 1} + \frac{nx^{2n+2}(3r - x)^{n-1}}{(2n + 1)(2n + 2)} + \frac{n(n - 1)x^{2n+3}(3r - x)^{n-2}}{(2n + 1)(2n + 2)(2n + 3)} + \dots + \frac{1.2.3.4 \dots n . x^{3n+1}}{(2n+1)(2n+2)\dots(3n+1)}\right]_0^{2r}$$

$$= \frac{2}{2n + 1} + \frac{2^2 n}{(2n + 1)(2n + 2)} + \frac{2^3 n(n - 1)}{(2n + 1)(2n + 2)(2n + 3)} + \dots + \frac{2^{n+1}(1.2.3 \dots n)}{(2n + 1)(2n + 2)\dots(3n + 1)}.$$

If $n = 2$, $p = \frac{26}{35}$; if $n = 3$, $p = \frac{46}{70}$, &c.

69.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Monroe County, Indiana.

Given the radius of Mars, 2250 miles, and the radius of the orbit of its inner satellite, 5800 miles; to determine whether the latter can have an elastic atmosphere, supposing its diameter to be 45 miles, and its density equal to that of the primary.

Remarks by the PROPOSER.

Soon after the discovery of Mars' satellites my attention was turned to the following passage in Laplace's System of the World: —

"If other bodies circulate round that which has been considered, or if it circulates itself round another body, the limit of its atmosphere is that point where its centrifugal force, *plus* the attraction of the extraneous body, exactly balances its gravity. Thus the limit of the moon's atmosphere is the point where the centrifugal force due to its rotary motion, *plus* the attractive force of the earth, is in equilibrium with the attraction of this satellite. The mass of the moon being $\frac{1}{75}$ of that of the earth, this point is therefore distant from the center of the moon about the ninth part of the distance of the moon from the earth. If, at this distance, the primitive atmosphere of the moon had not been deprived of its elasticity, it would have been carried towards the earth which might have retained it. This is perhaps the cause why this atmosphere is so little perceptible." *Book IV, Chap. X.*

This statement gives us the equation $\frac{M}{(a-x)^2}+v^2x=\frac{m}{x^2}$, where $M=$ the mass of the central body, $m=$ that of a satellite, $a=$ the distance between their centers, $x=$ the distance from the satellite to the limit of stability, $v=$ the satellite's velocity of rotation. Solving this equation for the earth and moon we obtain the result given in the foregoing extract. When we solve it, however, for Mars and its inner satellite we find the limit of stability *within the surface of the satellite.*

Hence, according to Laplace, the latter can have no atmosphere, nor can the body itself ever have existed in a fluid form. Can this result be accepted? If not, the statement of Laplace is erroneous. Let us inquire, then, what *is* the disturbing or divellent force of the earth on a particle of matter between itself and the moon, but in the vicinity of the latter? It is well known that the moon's attraction produces terrestrial tides by the *difference* between its force at the center and at the surface of the earth. The case would evidently be the same with the earth's action on the moon, or with Mars' action on its satellites. But this will give us a result very different from that obtained by the formula of Laplace. We conclude, therefore, that for anything we know to the contrary, either satellite of Mars may be surrounded by an atmosphere.

Note by Professor Peirce.

I have satisfied myself that the limits of atmosphere given by Laplace are incorrect; and that there is in reality no limitation which can be given to the hight of the atmosphere of a satellite.

70.—Proposed by John H. Adams, Cochranton, Crawford County, Pennsylvania.

In digging a well 6 feet in diameter a log 3 feet in diameter was found lying directly across the center of the well. How many cubic feet of the log must be removed from the well?

I.—Solution by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Put $R=3$ feet $=$ radius of the well and $r=1\frac{1}{2}$ feet $=$ radius of the log.

Let the part of the log that will be removed be intersected by a plane parallel to the axes of the log and well, and at a distance x from them, and $V=$ the required volume; then $4(R^2-x^2)^{\frac12}(r^2-x^2)^{\frac12}=$ area of the section, and $V=4\int_{-r}^{+r}(R^2-x^2)^{\frac12}(r^2-x^2)^{\frac12}dx.$

But

$$(R^2-x^2)^{\frac12}=R-\frac{x^2}{2R}-\frac{x^4}{8R^3}-\frac{x^6}{16R^5}-\frac{5x^8}{128R^7}-\frac{7x^{10}}{256R^9}-\frac{21x^{12}}{1024R^{11}}-\frac{33x^{14}}{2048R^{13}}-\dots-\frac{1.3.5.7\dots(2n-3)x^{2n}}{2.4.6\ 8\dots 2nR^{2n-1}}-\text{etc.}$$

Multiplying this by $4(r^2-x^2)^{\frac12}dx$ and integrating each term separately by the formula

$$\int_{-r}^{+r}x^{2n}(r^2-x^2)^{\frac12}dx=\left(\frac{1\ 3.5\dots(2n-1)}{2.4.6.8\dots(2n+2)}\right)\pi r^{2n+2},\text{ we have finally}$$

$$\text{V}=4r^2R\pi\left(\frac12-\frac{r^2}{16R^2}-\frac{r^4}{128R^4}-\frac{5r^6}{2048R^6}-\frac{35r^8}{32768R^8}-\frac{147r^{10}}{262144R^{10}}-\frac{693r^{12}}{2097152R^{12}}\right.$$
$$\left.-\frac{14157r^{14}}{67108864R^{14}}-\frac{[1.3.5.7\dots(2n-3)]^2(2n-1)r^{2n+1}}{[2.4\ 6.8.10\dots 2n]^2(2n+2)R^{2n+1}}-\text{etc.}\right),$$

$$=27\pi\left(\frac12-\frac{1}{64}-\frac{1}{2048}-\frac{5}{131072}-\frac{35}{8388608}-\frac{147}{268435456}-\frac{693}{8589934592}-\frac{14157}{1099511627776}-\text{etc.}\right).$$

$=41.041+$ cubic feet.

II.—Solution by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Put $r=$ radius of the log, and $R=$ radius of the well. Take the origin of rectangular co-ordinates at the intersection of the axes of the log and well, the axis of the log being the axis of y and the axis of the well the axis of z; then the equations to the surfaces of the log and well are

$$x^2+z^2=r^2\ \dots\dots(1),\qquad x^2+y^2=R^2\ \dots\dots(2).$$

Let $V=$ the volume removed; then $V=8\iiint dxdydz=8\iint ydxdz\ \dots\dots(3).$

The limits of y are 0 and $(R^2-x^2)^{\frac12}$; of z, 0 and $(r^2-x^2)^{\frac12}$; of x, 0 and r.

$$\therefore\quad V=8\iint(r^2-x^2)^{\frac12}\,dxdy=8\int_0^r(r^2-x^2)^{\frac12}(R^2-x^2)^{\frac12}\,dx\ \dots\dots(4).$$

Let $\frac{r}{R}=e$, and $x=rv$; then

$$V=8r^2R\int_0^1(1-v^2)^{\frac12}(1-e^2v^2)^{\frac12}\,dv,=8r^2R\int_0^1\frac{[1-(1+e^2)v^2+e^2v^4]dv}{(1-v^2)^{\frac12}(1-e^2v^2)^{\frac12}},$$
$$=\tfrac{8}{3}r^2R\Big[v(1-v^2)^{\frac12}(1-e^2v^2)^{\frac12}\Big]_0^1+\tfrac{8}{3}r^2R\int_0^1\frac{[2-(1+e^2)v^2]dv}{(1-v^2)^{\frac12}(1-e^2v^2)^{\frac12}},$$
$$=\tfrac{8}{3}R^3(1+e^2)\int_0^1\frac{(1-e^2v^2)^{\frac12}\,dv}{(1-v^2)^{\frac12}}-\tfrac{8}{3}R^3(1-e^2)\int_0^1\frac{dv}{(1-v^2)^{\frac12}(1-e^2v^2)^{\frac12}},$$
$$=\tfrac{8}{3}R^3\Big[(1+e^2)\text{E}(e)-(1-e^2)\text{F}(e)\Big].$$

71.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the United States Coast Survey, Cambridge, Middlesex County, Massachusetts.

Given the skill of two billiard players at the three-ball game, to find the chance of the better player gaining the victory if he gives the other a *grand discount*.—[From *Our Schoolday Visitor*, vol. xv, p 220.

Solution by the PROPOSER.

Let the two players be (A) and (B), whose probabilities of making a single shot are respectively a and b, and let (A) give the grand discount. (A) looses all he has made when (B) succeeds in making a shot.

Let $h=$ the number of points to be counted per game, $A=\frac{a}{a+b-ab}$, $B=\frac{b}{a+b-ab}$ so that $1-A=(1-a)B$, $1-B=(1-b)A$, $A-a=A(1-a)(1-b)$, $B-b=B(1-a)(1-b)$.

When (A) has i shots more to make and (B) has j shots more, let $F(i,j)=$ (A's) probability of winning when he has the play, and $f(i,j)=$ (A's) probability of winning when (B) has the play. The fundamental equations are, obviously, $F(i,j)=aF(i-1,j)+(1-a)\ f(i,j)$, $f(i,j)=bf(h,j-1)+(1-b)F(i\ j)$, which give by substitution, transposition and division $F(i,j)=A.F(i-1,j)+(1-A)f(h,j-1)$.

In special cases, we see that $F(i,0)=f(i,0)=0$, $F(1,1)=A$, $F(0,j)=1$, $F(i,1)=A.F(i-1,1)=A^i$; whence $F(i,j)-f(h,j-1)=A[F(i-1,j)-f(h,j-1)]=A^i[1-f(h,j-1)].=A^i+(1-A^i)f(h,j-1)$;

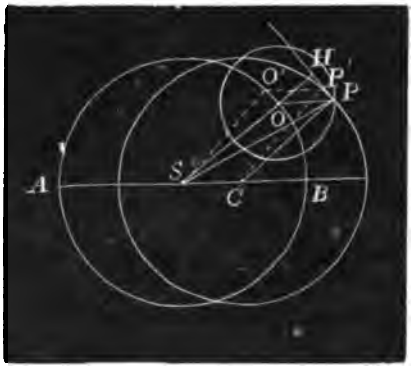
$F(i-1,j)=A^{i-1}+(1-A^{i-1})f(h,j-1)$, which, substituted in the first fundamental equation, gives $A^i(1-b)[1-f(h,j-1)]+f(h,j-1)=f(i,j)$; and if $c=A^h(1-b)$, $f(h,j)=(1-c)f(h,j-1)+c$,

$$1-f(h,j)=(1-c)[1-f(h,j-1)]=(1-c)^j;\ F(i,j)=A^i+(1-A^i)[1-(1-c)^{j-1}],$$
$$=1-(1-c)^{j-1}(1-A^i);\ F(h,h)=1-(1-c)^{h-1}(1-A^h).$$

72.—Proposed by DUNLAP J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Washington County, Pennsylvania.

The center of an epicycle whose radius is $\frac{1}{13}a$ revolves on a deferent, radius a, with uniform angular velocity v; a particle revolves in the epicycle so that the radius drawn in the small circle to the particle is always parallel to itself. Supposing the particle to be moving under the influence of a force at the center of the large circle, find the law of the force.

Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.



Let S be the center of the deferent, and a its radius, O the center of the epicycle and P the position of the particle at any time, ASB the diameter of the deferent to which the radius drawn in the epicycle to the particle is always parallel. On SB take SC = OP, join SO, OP, PC and SP; SOPC is a parallelogram, CP = SO, hence the path of the particle is a circle, center C and radius a. Let O′, P′ be the next consecutive positions of O and P; SO′, CP′ being parallel, angle OSO′ = angle PCP′; PP′ = OO′, and the particle moves with the uniform velocity that the center of the epicycle moves on the deferent.

Supposing the particle to move under the influence of a center of force at S, draw a tangent to its path at P, produce SO to meet the tangent in H; SH is the perpendicular from the center of force on the tangent, which put $=p$; and put SP $=r$, OP $=b$. Then $\cos OSP=\frac{a^2-b^2+r^2}{2ar}$, $SH=r\cos OSP$;

$\therefore\ p=\frac{a^2-b^2+r^2}{2a}$. For the velocity we have the well-known formula, $v=\frac{h}{p}$, h being a constant.

$\therefore\ v=\frac{2ah}{a^2-b^2+r^2}$, or, $2ah$ being constant, the velocity varies inversely as $a^2-b^2+r^2$, and the given velocity being constant, the supposition that the particle moves under the influence of a center of force at S with the given velocity is impossible. Supposing it possible for the particle to move with the velocity found for a center of force at S, to find the law of the force we have the well-known formula, $F=\frac{h^2}{p^3}\cdot\frac{dp}{dr}$. Substituting the value of p we find $F=\frac{8a^2h^2r}{(a^2-b^2+r^2)^3}$, or, $8a^2h^2$ being constant, the force varies inversely as $\frac{(a^2-b^2+r^2)^3}{r}$.

NOTE BY THE PROPOSER.—Problem 72 gives the conditions involved in the Hipparchian method of representing the moon's motion. Hipparchus imagined the moon to move with uniform velocity in a circle of which the earth occupied a point at a distance of $\frac{1}{13}$ radius from the center. Another method of considering the motion was by means of an epicycle such as given in the problem. He constructed his system so as to "save the observations." If he had constructed a system which would save the laws of motion of a body under a central force, he would have left out the condition of "uniform velocity." His entire theory would then have been abandoned.

73.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Find the average distance between two points taken at random in the surface of a given semicircle.

I.—Solution by E. B. Seitz, Greenville, Darke County, Ohio.

Let ACB be the given semicircle, O its center, M, N two points such that the line through them intersects the arc of the semicircle in two points C, D, and M′, N′ two points such that the line through them is parallel to CD, and intersects the arc of the semicircle in C′ and the base in D′. Draw OH perpendicular to CD.

Let $OA = r$, CM or $C'M' = x$, MN or $M'N' = y$, $CD = u$, $C'D' = v$, $\angle AOH = \theta$, $\angle COH$ or $C'OH = \varphi$.

Then $u = 2r\sin\varphi$, $v = r(\sin\varphi + \tan\theta\cos\varphi)$; an element of the semicircle at M or M′ is $r\sin\varphi d\varphi dx$, at N or N′ it is $d\theta y dy$. The limits of θ are 0 and $\frac{1}{2}\pi$, and doubled; those of x are 0 and u, when $\varphi < \theta$, and 0 and v, when $\varphi > \theta$; those of y are 0 and x, and doubled; and those of φ are 0 and θ, and θ and $\pi - \theta$.

Hence, since the whole number of ways the two points can be taken is $\frac{1}{4}\pi^2 r^4$, the required average is

$$\Delta = \frac{16}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\left(\int_0^{\theta}\int_0^{u}\int_0^{x} r\sin\varphi d\varphi dx y^2 dy + \int_{\theta}^{\pi-\theta}\int_0^{v}\int_0^{x} r\sin\varphi d\varphi dx y^2 dy\right)d\theta,$$

$$= \frac{16}{3\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\left(\int_0^{\theta}\int_0^{u} x^3\sin\varphi d\varphi dx + \int_{\theta}^{\pi-\theta}\int_0^{v} x^3\sin\varphi d\varphi dx\right)d\theta,$$

$$= \frac{4r}{3\pi^2}\int_0^{\frac{1}{2}\pi}\left(\int_0^{\theta} 16\sin^5\varphi d\varphi + \int_{\theta}^{\pi-\theta}(\sin\varphi + \tan\theta\cos\varphi)^4\sin\varphi d\varphi\right)d\theta,$$

$$= \frac{64r}{45\pi^2}\int_0^{\frac{1}{2}\pi}(8 - 7\cos\theta - 2\sin^2\theta\cos\theta)d\theta = \frac{256r}{45\pi} - \frac{1472r}{135\pi^2}.$$

II.—Solution by the Proposer.

Let P and Q be the random points, P being nearer than Q to AO, $OQ = x$, $OP = y$, $\angle AOP = \varphi$, $\angle AOQ = \theta$, $PQ = z$ and Δ = average distance required. Then $z = [x^2 + y^2 - 2xy\cos(\theta - \varphi)]^{\frac{1}{2}}$, and

$$\Delta = \frac{\int_0^{\pi}\int_0^{\theta}\int_0^{r}\int_0^{x} zxdxydyd\varphi d\theta}{\int_0^{\pi}\int_0^{\theta}\int_0^{r}\int_0^{x} xdxydyd\varphi d\theta} = \frac{16}{\pi^2 r^4}\int_0^{\pi}\int_0^{\theta}\int_0^{r}\int_0^{x}[x^2 + y^2 - 2xy\cos(\theta - \varphi)]^{\frac{1}{2}}xdxydyd\varphi d\theta,$$

$$= \frac{8}{3\pi^2 r^4}\int_0^{\pi}\int_0^{\theta}\int_0^{r}[16\sin^3\tfrac{1}{2}(\theta - \varphi) + 12\sin^3\tfrac{1}{2}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi)$$
$$+ 3\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log\{1 + \operatorname{cosec}\tfrac{1}{2}(\theta - \varphi)\}]x^4 dx d\theta d\varphi,$$

$$= \frac{8r}{15\pi^2}\int_0^{\pi}\int_0^{\theta}[16\sin^3\tfrac{1}{2}(\theta - \varphi) + 12\sin^3\tfrac{1}{2}(\theta - \varphi)\cos(\theta - \varphi) - 2 + 3\cos^2(\theta - \varphi)$$
$$+ 3\sin^2(\theta - \varphi)\cos(\theta - \varphi)\log\{1 + \operatorname{cosec}\tfrac{1}{2}(\theta - \varphi)\}]d\theta d\varphi,$$

$$= \frac{4r}{45\pi^2}\int_0^{\pi}[48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta + 12\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 32\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta - 6\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$
$$- 64\cos\tfrac{1}{2}\theta + 64 + 9\sin\theta\cos\theta + 6\sin^3\theta\log(1 + \operatorname{cosec}\tfrac{1}{2}\theta)]d\theta,$$

$$= \frac{256r}{45\pi} - \frac{1472r}{135\pi^2}.$$

This problem was also solved in an elegant manner by *Henry Heaton*.

74.—Proposed by Rev. W. J. Wright, M. A., Ph. D., Member of the London Mathematical Society, Carlisle, Cumberland County, Pennsylvania.

Transform the Eulerian integral $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$ to the Legendreian $\Gamma(n) = \int_0^1\left(\log\frac{1}{x}\right)^{n-1}dx$.

Solution by Prof. D. J. McAdam; Prof. H. T. J. Ludwig; Prof. F. P. Matz; and E. B. Seitz.

Let $e^{-x} = y$, then $-x = \log y$, or $x = \log\left(\frac{1}{y}\right)$ and $dx = -\frac{dy}{y}$. For $x = \infty$, $y = 0$; for $x = 0$, $y = 1$; therefore the limits must now be 0 and 1. Substituting, $\Gamma(n) = -\int_1^0\left(\log\frac{1}{y}\right)^{n-1}dy = \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$.

Since the value of the last result is independent of the name of the variable we may write x for y, which gives the result required.

75.—Proposed by Prof. H. A. Wood, M. A., Principal Coxsackie Academy, Coxsackie, Greene County, New York.

A sphere 4 inches in diameter, specific gravity 0.2, is placed 10 feet under water. If left free to move, what will be its velocity at the surface of the water, and what will be the maximum hight it will attain?

Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let $r=\frac{1}{6}$ = radius of the sphere, $\rho=\frac{1}{6}$ = its specific gravity, $h=10$ feet, v = the velocity acquired in ascending a distance x, V = the velocity at the surface of the water, k = the resistance.

For the motion of the sphere, $\frac{vdv}{dx}=g\left(\frac{1}{\rho}-1\right)-kv^2$. Putting $g\left(\frac{1}{\rho}-1\right)=g'$, $dx=\frac{vdv}{g'-kv^2}$.

$$\int_0^h dx=\int_0^V\frac{vdv}{g'-kv^2},\ h=\frac{1}{2k}\log\left(\frac{g'}{g'-kV^2}\right);\ \text{whence } V^2=\frac{g'}{k}\left(1-e^{-2hk}\right)=\frac{g(1-\rho)}{k\rho}\ \text{nearly.}$$

The required hight $=\frac{V^2}{2g}=\frac{1-\rho}{2k\rho}\left(1-e^{-2hk}\right)=\frac{1-\rho}{2k\rho}$ nearly.

The value of k deduced from Newton's Principia, Book 2, Prop. 38, and used by Atwood, Hutton, Simpson, Emerson, and all English authors until recently is $\frac{3}{16r\rho}$. This value of k gives $V=[\frac{16}{3}rg(1-\rho)]^{\frac{1}{2}}$ $=\frac{1}{3}\sqrt{(6g)}$ nearly, and $h=\frac{V^2}{2g}=\frac{8r(1-\rho)}{3}=\frac{10}{27}=4\frac{4}{9}$ inches nearly.

Poisson rejects Newton's Theory of Resistances, and in Vol. 2, Art. 367 of his Mechanics gives an approximate value of $k=\frac{9}{40r\rho}$. This value of k gives $h=3\frac{2}{3}$ inches. Whewell, Earnshaw and Tait do not give any method for finding the numerical value of k, and Rankine in his *Applied Mechanics*, p. 599, gives for the resistance in a fluid, $R=\frac{keAv^2}{2g}$, where e is the weight of a unit of the fluid, A the area of the greatest cross-section, and k for a sphere $=0.51$.

Solved also by Prof. *De Volson Wood*, who, by using the coefficient of resistance given by Rankine, finds $h=0.338+$ feet, $=4.056$ inches

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76.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics Johns Hopkins University, Baltimore, Maryland.

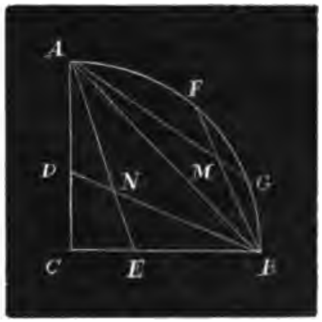
Prove that the equation to the sums and the equation to the products of the roots of an equation of the nth degree taken two and two together may each be put under the form of a determinant of the order $\frac{1}{2}(n-1)$ or of the order $\frac{1}{2}n$ according as n is odd or even, and write down the determinant of the third order which equated to zero is the equation (of the degree 21) to the binary products of an equation of the 7th degree clear of any irrelevant factor.

[We have received no solution of this problem. We hope to receive one for publication in No. 4.]

77.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Two points are taken at random in the surface of a quadrant of a circle, and a line drawn through them. Find the chance (1) that the line intersects the arc in two points, (2) that it intersects the arc in one point, and (3) that it does not intersect the arc.

Solution by HENRY HEATON, Principal of Public Schools, Sabula, Jackson County, Iowa.



Let p_1, p_2 and p_3 equal the required probabilities in the order named. Then it is evident that $p_1+p_2+p_3=1$. To find p_1 let M in the figure be one of the points, then the other must be either on the surface MGB or on AMF when the line intersects the arc in two points. Put AC = 1, AM = x, $\angle$MAB = θ, surface MGB = u and surface AMF = v. It is evident that the sum of all values of u equals the sum of all values of v. Hence we may use $2u$ instead of $u+v$. $u=\theta-\sin^2\theta-\frac{1}{2}\sqrt{2}x\sin\theta$.

The probability that the first point falls on the element of surface at M is $\frac{xdxd\theta}{\frac{1}{4}\pi}$. The probability that the second point falls on either u or v is $\frac{u+v}{\frac{1}{4}\pi}$. The limits of x are 0 and $2\cos(\frac{1}{4}\pi+\theta)=x'$; of θ, 0 and $\frac{1}{4}\pi$.

Hence $p_1=\frac{16}{\pi^2}\int_0^{\frac{1}{4}\pi}\int_0^{x'}2uxdxd\theta$

$$=\frac{32}{3\pi^2}\int_0^{\frac{1}{4}\pi}[3\theta(1-2\sin\theta\cos\theta)-\sin^2\theta-2\sin\theta\cos^3\theta+4\sin^2\theta\cos^2\theta]d\theta=1-\frac{28}{3\pi^2}.$$

To find p_3, let N be one of the points, then if the line does not intersect the arc, the other point lies either on the surface ENB or on AND.

Put AN = y, $\angle$CAN = φ, $\triangle$ENB = u_1, and $\triangle$AND = v_1. Then, as before, $2u_1$ may be used instead of u_1+v_1. $u_1=\frac{1}{2}-\frac{1}{2}\tan\varphi-\frac{1}{2}\sqrt{2}y\sin(\frac{1}{4}\pi-\varphi)$. The limits of y are 0 and $\sec\varphi=y'$; of φ, 0 and $\frac{1}{4}\pi$.

Hence $p_3=\frac{16}{\pi^2}\int_0^{\frac{1}{4}\pi}\int_0^{y'}2u_1ydyd\varphi=\frac{8}{3\pi^2}\int_0^{\frac{1}{4}\pi}(\sec^2\varphi-\tan\varphi\sec^2\varphi)d\varphi=\frac{4}{3\pi^2}$. $p_2=1-p_1-p_3=\frac{8}{\pi^2}$.

This problem was also solved very elegantly by the *Proposer*.

78.—Proposed by Dr Volson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

One end of a fine, inextensible string is attached to a fixed point, and the other end to a point in the surface of a homogeneous sphere, and the ends brought together, the center of the sphere being in a horizontal through the ends of the string, and the slack string hanging vertically. The sphere is let fall and an angular velocity imparted to it at the same instant, the sphere winding up the string on the circumference of a great circle until it winds up all the slack when it suddenly begins to ascend, winding up the string, the sphere returning just to the starting point. Required the initial angular velocity, the tension of the string during the ascent of the sphere, the initial upward velocity of the center of the sphere, and the time of the movement.

Solution by Walter Siverly, Oil City, Venango County, Pennsylvania; and the Proposer.

Let $l =$ the length of the string, $r =$ radius of the sphere, $m =$ its mass, $k =$ its radius of gyration, $x =$ the length of the unwound portion of string at the end of descent, $v =$ velocity at the end of descent, $v' =$ velocity with which the sphere begins to ascend, $\omega =$ the initial angular velocity, $\omega' =$ the angular velocity with which the sphere begins to ascend, $t_1 =$ time of descent, $t_2 =$ time of ascent, $T =$ tension of the string during the ascent and $I =$ the impulse communicated by the string to the sphere at the end of descent.

The length of string wound up at the end of descent $= r\omega t_1 = r\omega\sqrt{\left(\frac{2x}{g}\right)}$, t_1 being the time of falling freely through the hight x. $\therefore\ x = l - r\omega\sqrt{\left(\frac{2x}{g}\right)}$, whence $\omega = \frac{l-x}{r}\sqrt{\left(\frac{g}{2x}\right)}$ ....(1).

For the impulsive motion, $\omega - \omega' = \frac{Ir}{mk^2}$, $v + v' = \frac{I}{m}$. Eliminating I, $\frac{k^2}{r}(\omega - \omega') = v + v'$.

But $k^2 = \frac{2}{5}r^2$, $v = \sqrt{(2gx)}$; $\therefore\ \omega' = \omega - \frac{5}{2r}[\sqrt{(2gx)} + v']$ ....(2).

For the upward motion, the origin being at the point where the center begins to ascend and the axis of y positive upwards, $m\frac{d^2y}{dt^2} = T - mg$ ....(3).

For the angular acceleration, $mk^2\frac{d^2\theta}{dt^2} = -rT$, or $m\frac{d^2\theta}{dt^2} = -\frac{5T}{2r}$...(4). Also $y = r\theta$ or $\frac{d^2y}{dt^2} = \frac{rd^2\theta}{dt^2}$ ...(5).

Eliminating $\frac{d^2y}{dt^2}$, $\frac{d^2\theta}{dt^2}$ from (3), (4), (5), $T = \frac{2}{7}mg = \frac{2}{7}$ the weight of the sphere. Eliminating T, $\frac{d^2\theta}{dt^2}$ from (3), (4), (5), $\frac{d^2y}{dt^2} = -\frac{5}{7}g$. Integrating, observing that when $t = 0$, $\frac{dy}{dt} = v'$, $\frac{dy}{dt} = v' - \frac{5}{7}gt$............(6). When $\frac{dy}{dt} = 0$, $t = t_2$; $\therefore\ v' = \frac{5}{7}gt_2$...(7). Integrating (6), observing that when $t = 0$, $y = 0$, $y = v't - \frac{5}{14}gt^2$.

When $y = x$, $t = t_2$; $\therefore\ x = v't_2 - \frac{5}{14}gt_2^2$...(8). From (7), (8) $v' = \sqrt{(\frac{10}{7}gx)}$......(9), $t_2 = \sqrt{\left(\frac{14x}{5g}\right)}$... (10).

From (5) $\frac{dy}{dt} = r\frac{d\theta}{dt}$, $\therefore\ v' = r\omega'$, $\omega' = \frac{1}{r}\sqrt{(\frac{10}{7}gx)}$,.......(11). Substituting these values of v', ω' in (2), $x = l(6-\sqrt{35})$. Substituting this value of x in (1), (9), $\omega = \frac{[\sqrt{(35)}-5]\sqrt{(gl)}}{r\sqrt{\{2[6-\sqrt{(35)}]\}}}$, $v' = \sqrt{\{\frac{10}{7}gl[6-\sqrt{(35)}]\}}$.

$t_1 = \sqrt{\left(\frac{2x}{g}\right)} = \sqrt{\left(\frac{2l[6-\sqrt{(35)}]}{g}\right)}$. From (10), $t_2 = \sqrt{\left(\frac{14x}{5g}\right)} = \sqrt{\left(\frac{14l[6-\sqrt{(35)}]}{5g}\right)}$.

The whole time $= t_1 + t_2 = (1 + \frac{1}{5}\sqrt{35})\sqrt{\left(\frac{2l[6-\sqrt{(35)}]}{g}\right)}$.

This problem was also solved by Prof. *F. P. Matz*.

79.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Find the average distance of the center of an ellipse, axes $2a$ and $2b$, from its circumference.

Solution by Henry Heaton, Principal Public Schools, Sabula, Jackson County, Iowa.

Let $\Delta =$ average distance required; then $\Delta = \frac{\int r\,dz}{\int dz}$, where r is the distance from the center to any point P in the circumference of the ellipse, and dz is the differential of the arc estimated from the extremity of the minor axis.

Let x and y be co-ordinates of the point P; then the equation of the ellipse is $a^2y^2 + b^2x^2 = a^2b^2$, $r = \sqrt{(x^2+y^2)}$ and $dz = \sqrt{(dx^2+dy^2)}$. But $y = \frac{b}{a}\sqrt{(a^2-x^2)}$, therefore $r = \sqrt{(b^2+e^2x^2)}$, putting $e^2 = \frac{a^2-b^2}{a^2}$, and $dz = \frac{\sqrt{(a^2-e^2x^2)}dx}{\sqrt{(a^2-x^2)}}$.

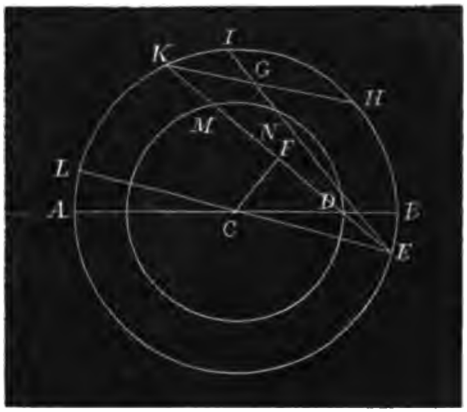
Substituting, $\Delta=\dfrac{1}{a\mathrm{E}(e,\,\frac{1}{2}\pi)}\displaystyle\int_0^a\frac{\sqrt{(b^2+e^2x^2)}\sqrt{(a^2-e^2x^2)}dx}{\sqrt{(a^2-x^2)}}$. Assume $x^2=\frac{1}{2}a^2+\frac{1}{2}a^2\sin\theta$, then

$$\Delta=\frac{1}{4a\mathrm{E}(e,\,\frac{1}{2}\pi)}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sqrt{[(a^2+b^2)^2-a^4e^4\sin^2\theta]}d\theta,=\frac{(a^2+b^2)\mathrm{E}\left(\frac{a^2-b^2}{a^2+b^2},\,\frac{1}{2}\pi\right)}{2a\mathrm{E}(e,\,\frac{1}{2}\pi)}=\frac{a(2-e^2)\mathrm{E}\left(\frac{e^2}{2-e^2},\,\frac{1}{2}\pi\right)}{2\mathrm{E}(e,\,\frac{1}{2}\pi)}.$$

30.—Proposed by E. B. Seitz, Greenville, Darke County, Ohio.

A chord is drawn through two points taken at random in the surface of a circle. If a second chord be drawn through two other points taken at random in the surface, find the chance that it will intersect the first chord.

I.—Solution by Henry Heaton, Principal of Public Schools, Sabula, Jackson County, Iowa.



Let EK be the first chord, drawn through the points D and N. Let G be one of the two other points. Put $BC=1$, $CD=x$, $DN=y$, $EG=z$, $\angle CDN=\theta$, $\angle CEG=\varphi$, $\angle CEK=\psi=\sin^{-1}(x\sin\theta)$, surface $KGI=u$; surface $EGH=v$, required probability $=p$, and probability the two chords will *not* intersect $=p'$.

Then $u=\varphi-\psi-\frac{1}{2}\sin 2\psi+\frac{1}{2}\sin 2\varphi-z\cos\psi\sin(\varphi-\psi)$. If the fourth point falls upon either u or v the chords will not intersect. If G occupies every point of the segment KEBHI, the sum of all values of v equals the sum of all values of u, hence in taking the sum $2u$ may be used for $u+v$.

Confining G to one side of the chord EK, and N within the circle through D, will evidently each give half the result obtained by allowing them to occupy every point of the circle.

The probabilities, that x will have any particular value, that N will fall upon the element of surface at N, that G will fall upon the element of surface at G, and that the fourth point will fall upon u or v, are respectively $\dfrac{2\pi xdx}{\pi}$, $\dfrac{ydyd\theta}{\pi}$, $\dfrac{zdzd\varphi}{\pi}$ and $\dfrac{u+v}{\pi}$.

The limits of z are 0 and $2\cos\varphi,=z'$; of φ, ψ and $\frac{1}{2}\pi$; of y, 0 and $2x\cos\theta=y'$; of x, 0 and 1; of θ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

Hence $p'=\dfrac{16}{\pi^3}\displaystyle\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^1\int_0^{y'}\int_\psi^{\frac{1}{2}\pi}\int_0^{z'}ud\theta xdxydyd\varphi zdz,$

$$=\frac{16}{3\pi^3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^1\int_0^{y'}\int_\psi^{\frac{1}{2}\pi}[3\cos^2\varphi(2\varphi-2\psi-\sin2\psi+\sin2\varphi)-8\cos\psi\cos^3\varphi\sin(\varphi-\psi)]d\theta xdxydyd\varphi,$$

$$=\frac{2}{3\pi^3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^1\int_0^{y'}(3\pi^2-12\pi\psi+\psi^2-16\cos^2\psi+4\sin^2\psi\cos^2\psi)d\theta xdxydy,$$

$$=\frac{4}{3\pi^3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^1(3\pi^2-12\pi\psi+12\psi^2-16\cos^2\psi+4\sin^2\psi\cos^2\psi)\cos^2\theta d\theta x^3dx,$$

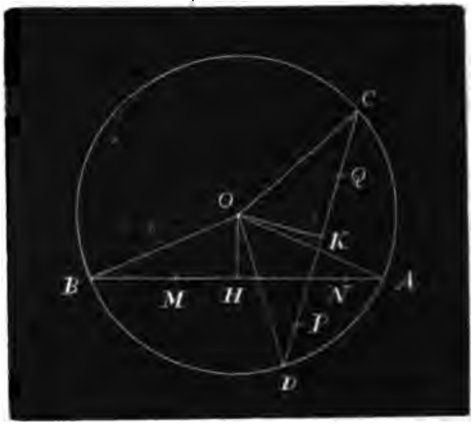
$$=\frac{4}{3\pi^3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}\int_0^\theta(3\pi^2-12\pi\psi+12\psi^2-16\cos^2\psi+4\sin^2\psi\cos^2\psi)\cos^2\theta\,\mathrm{cosec}^4\theta\sin^3\psi\cos\psi d\theta d\psi,$$

$$=\frac{1}{18\pi^3}\int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi}[72\theta^2\cos^2\theta-27\theta^2\cos^2\theta\,\mathrm{cosec}^4\theta+36\theta\cos^2\theta\cot\theta+54\theta\cot^3\theta-72\pi\theta\cos^2\theta$$

$$+27\pi\theta\cos^2\theta\,\mathrm{cosec}^4\theta-18\pi\cos^2\theta\cot\theta-27\pi\cot^3\theta-27\cot^2\theta+(18\pi^2-37)\cos^2\theta-56\cos^4\theta-12\cos^6\theta]d\theta$$

$$=\frac{2}{3}-\frac{245}{72\pi^2}.\quad\text{But } p+p'=1,\quad\therefore\quad p=\frac{1}{3}+\frac{245}{72\pi^2}.$$

II.—Solution by the Proposer.



Let M, N be the first two random points, AB the chord through them, P, Q the second two random points, CD the chord through them and O the center of the circle. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $AM = w$, $MN = x$, $CP = y$, $PQ = z$, $\angle AOH = \theta$, $\angle COK = \varphi$, $\angle AOC = \psi$, and $\mu =$ the angle AB makes with some fixed line.

Then we have $AH = r\sin\theta$, $CK = r\sin\varphi$; an element of the circle at M is $r\sin\theta d\theta dw$; at N, $d\mu xdx$; at P, $r\sin\varphi d\varphi dy$; at Q, $d\psi zdz$. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and doubled; of ψ, 0 and 2φ, and doubled; of μ, 0 and 2π; of w, 0 and $2r\sin\theta, = w'$; of x, 0 and w, and doubled; of y, 0 and $2r\sin\varphi, = y'$; and of z, 0 and y, and doubled. Hence, since the whole number of ways the four points can be taken is π^4r^8, the required chance is

$$p = \frac{16}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'}\int_0^{y} r\sin\theta d\theta r\sin\varphi d\varphi d\psi d\mu dw x dx dy z dz,$$

$$= \frac{8}{\pi^4r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'} \sin\theta\sin\varphi d\theta d\varphi d\psi d\mu dw x dx y^2 dy,$$

$$= \frac{64}{3\pi^4r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w} \sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu dw x dx,$$

$$= \frac{32}{3\pi^4r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'} \sin\theta\sin^4\varphi d\theta d\varphi d\psi d\mu w^2 dw, = \frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi} \sin^4\theta\sin^4\varphi d\theta d\varphi d\psi d\mu,$$

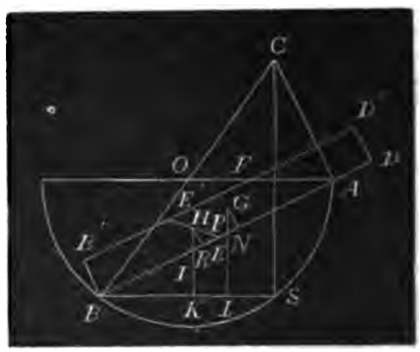
$$= \frac{512}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi} \sin^4\theta\sin^4\varphi d\theta d\varphi d\psi, = \frac{1024}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta} \varphi\sin^4\theta\sin^4\varphi d\theta d\varphi,$$

$$= \frac{64}{9\pi^3}\int_0^{\frac{1}{2}\pi}(8\theta^2 - 6\theta\sin\theta\cos\theta - 4\theta\sin^3\theta\cos\theta + 3\sin^2\theta + \sin^4\theta)\sin^4\theta d\theta, = \frac{1}{3} + \frac{245}{72\pi^2}.$$

81.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

A fixed hemispherical bowl, radius R, is full of water and contains a heavy cylindric rod, length $2a$, radius r and specific gravity ρ, having one end against its concave surface, and resting on its rim. Determine the inclination of the rod.

Solution by E. B. Seitz, Greenville, Darke County, Ohio.



Let O be the center of the bowl, BDD'B' a vertical section of the rod through its axis, A the point at which the rod touches the rim, B the point at which it touches the concave surface, G the center of gravity of the rod. Join E and E', the middle points of AB and B'F; then the center of gravity of the immersed portion of the rod is at some point of EE'; let H be the point. Draw AC perpendicular to AB, meeting BO produced in C; draw the vertical line CS, and join BS. Draw HK and GL perpendicular to BS, and HR and GN perpendicular to AB.

Let $BN = ND = a$, $AO = R$, $BB' = 2r$, $AB = m$, $B'F = n$, $\angle OAB = \angle ABS = \theta$, the inclination of the rod, and $\tan\frac{1}{2}\theta = t$. Then we have $m = 2R\cos\theta$, $n = 2R\cos\theta - 2r\cot\theta$, $\angle BCS = 90° - 2\theta$, $BS = 2R\cos 2\theta$, $BL = a\cos\theta - r\sin\theta$, $LS = 2R\cos 2\theta - a\cos\theta + r\sin\theta$,

$$HR = \frac{r(3m+5n)}{4(m+n)} = \frac{8Rr\cos\theta - 5r^2\cot\theta}{8R\cos\theta - 4r\cot\theta}, \quad RE = \frac{8Rr\cos\theta\cot\theta - 5r^2\cot^2\theta}{16R\cos\theta - 8r\cot\theta},$$

$$BR = \frac{16R^2\cos^2\theta - 16Rr\cos\theta\cot\theta + 5r^2\cot^2\theta}{16R\cos\theta - 8r\cot\theta}, \quad IR = \frac{8Rr\sin\theta - 5r^2}{8R\cos\theta - 4r\cot\theta},$$

$$\text{BK}=\frac{16R^2\sin^2\theta\cos^2\theta-16Rr\sin\theta+5r^2\sin^2\theta+5r^2}{16R\sin^2\theta-8r\sin\theta},\quad \text{KS}=\frac{32Rr\sin^2\theta+(16R^2-5r^2)\sin^2\theta-48R^2\sin^4\theta-5r^2}{16R\sin^2\theta-8r\sin\theta}.$$

The forces acting on the rod are a downward force at $\text{G}=2\pi ar^2\rho$, an upward force at $\text{H}=\frac{1}{2}\pi r^2(m+n)=\pi r^2(2R\cos\theta-r\cot\theta)$, and the reactions of the bowl at A and B, which forces meet at C. Taking moments of the forces at G and H about C, we have $\text{LS}\,.\,2\pi ar^2\rho=\text{KS}\,.\,\pi r^2(2R\cos\theta-r\cot\theta)$.

Substituting the values of LS and KS, and transforming the equation, we find

$$5r^3t^{12}-20R^2t^{11}-(128aRr\rho+64a^2r\rho+64R^2r-40r^3)t^{10}+(512aR^2\rho-128ar^2\rho+256a^2R\rho+256R^3-896Rr^2)t^9$$
$$+(1024aRr\rho+1664R^2r-128a^2r\rho+65r^3)t^8-(2560aR^2\rho+384ar^2\rho-256a^2R\rho+2816R^3+376Rr^2)t^7$$
$$+2304arR\rho t^6-(2560aR^2\rho+384ar^2\rho+256a^2R\rho-2816R^3-376Rr^2)t^5+(1024aRr\rho-1664R^2r+128a^2r\rho-65r^3)t^4$$
$$+(512aR^2\rho-128ar^2\rho-256a^2R\rho-256R^3+896Rr^2)t^3-(128aRr\rho-64a^2r\rho-64R^2r+40r^3)t^2+20Rr^2t-5r^3=0,$$

an equation of the 12th degree for the determination of θ.

82.—Proposed by John M. Wilt, M. A., Fort Wayne, Allen County, Indiana.

An unknown cone is cut at random by a plane; find the chance that the section is an ellipse.—[From the *Normal Monthly*, vol iii, p. 108.

Solution by Henry Heaton, Principal of Public Schools, Sabula, Jackson County, Iowa.

Put $a=$ altitude of the cone, $2\theta=$ angle at the vertex, $\varphi=$ angle the cutting plane makes with the axis of the cone, $p=$ the chance that the section is *elliptical* and q, that it is a *complete* ellipse, supposing a and θ known; and $\text{P}=$ the chance that the section is elliptical, and Q, that it is a complete ellipse, supposing all values of a and θ equally probable. The number of planes that can cut the axis at the angle φ is proportional to $\cos\varphi$.

If $\varphi<\theta$ the number of planes parallel to each other that can cut the cone is proportional to $2a\tan\theta\cos\varphi$, the distance between the extreme parallel planes that can cut it.

If $\varphi>\theta$ the section is *elliptical* and the number of parallel planes that can cut the cone is proportional to $a\sec\theta\sin(\varphi+\theta)$. In this case if the plane cuts both sides of the cone the section is a *complete* ellipse. The number of parallel planes that can do this is proportional to $a\sec\theta\sin(\varphi-\theta)$.

$$\text{Hence } p=\frac{\int_\theta^{\frac{1}{2}\pi}a\sec\theta\sin(\varphi+\theta)d\varphi}{\int_0^\theta 2a\tan\theta\cos^2\varphi d\varphi+\int_\theta^{\frac{1}{2}\pi}a\sec\theta\sin(\varphi+\theta)d\varphi},=\frac{(\frac{1}{2}\pi-\theta)\tan\theta+\cos2\theta}{(\frac{1}{2}\pi+\theta)\tan\theta+1},=1-\frac{2\theta+\sin2\theta}{\cot\theta+\theta+\frac{1}{2}\pi};$$

$$\text{and } q=\frac{\int_\theta^{\frac{1}{2}\pi}a\sec\theta\sin(\varphi+\theta)d\varphi}{\int_0^\theta 2a\tan\theta\cos^2\varphi d\varphi+\int_\theta^{\frac{1}{2}\pi}a\sec\theta\sin(\varphi+\theta)d\varphi},=\frac{1-(\frac{1}{2}\pi-\theta)\tan\theta}{1+(\frac{1}{2}\pi+\theta)\tan\theta}=1-\frac{\pi}{\cot\theta+\theta+\frac{1}{2}\pi}.$$

$$\text{P}=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}pd\theta=1-\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{(2\theta+\sin2\theta)d\theta}{\cot\theta+\theta+\frac{1}{2}\pi}.\quad \text{Q}=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}qd\theta=1-2\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\cot\theta+\theta+\frac{1}{2}\pi}.$$

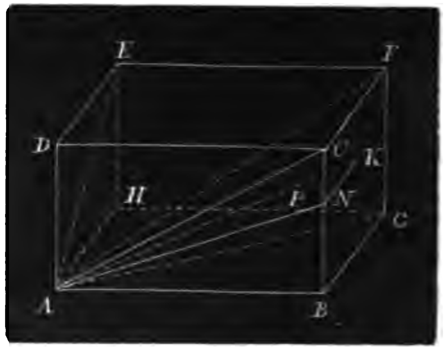
I have not been able to integrate these expressions, but by taking the ordinates 5° apart and applying Mr. Weddle's formula for approximate quadrature, $\int_0^{6\theta}ydx=\frac{3\theta}{10}[y_0+y_2+y_4+y_6+5(y_1+y_3+y_6)+y_3]$, to each 30°, I get $\int_0^{\frac{1}{2}\pi}\frac{(2\theta+\sin2\theta)d\theta}{\cot\theta+\theta+\frac{1}{2}\pi}=0.3231\pi$ and $\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\cot\theta+\theta+\frac{1}{2}\pi}=0.12922\pi$. Hence $\text{P}=0.3538$ and $\text{Q}=0.1880$.

In a solution of this problem in the *Lady's and Gentleman's Diary* for 1861, given by the Editor, the values of p and q are the same as found above; but the values of P and Q are found by integrating the numerator and denominator respectively, thus getting $\text{P}=\frac{\pi-\frac{4}{3}}{\pi+4}=0.2532$, and $\text{Q}=\frac{4-\pi}{4+\pi}=0.1202$. This method is evidently incorrect, since the number of planes whose section is an ellipse is divided by the whole number of cutting planes, all planes *not being equally probable.* It is the evident intention of the author of the problem that all *cones* and also all *planes* that cut the same cone, shall be equally probable. Hence it is evident that the probability for any given plane depends on the number of planes that can cut the cone through which it passes. Consequently all planes are not equally probable, and the solution referred to above is incorrect.

88.—Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

Find the average distance of one corner of a rectangular solid, edges a, b, c, from all points within it.

I.—Solution by E. B. Seitz, Greenville, Darke County, Ohio.



Let AF be the rectangular solid, P any point within it. From A draw AF, the diagonal of the solid, and AC, AG, AE, the diagonals of the faces. The solid consists of the three pyramids, ABCFG, ACDEF, AEFGH. Produce AP to K in the face BCFG, and draw KN perpendicular to BC. Let $AB=a$, $AH=b$, $AD=c$, $AF=e$, $AG=e_1$, $AC=e_2$, $AE=e_3$, $AP=x$, $AK=x'$, $\angle BAN=\theta$, $\angle PAN=\varphi$, $\varphi'=\tan^{-1}\left(\frac{b\cos\theta}{a}\right)$, $\theta'=\tan^{-1}\left(\frac{c}{a}\right)$, and let $\Delta=$ the required average, $\Delta_1=$ the average distance of A from all points within the pyramid ABCFG, $\Delta_2=$ the average distance of A from all points within the pyramid ACDEF, and $\Delta_3=$ the average distance of A from all points within the pyramid AEFGH.

Then $e=\sqrt{(a^2+b^2+c^2)}$, $e_1=\sqrt{(a^2+b^2)}$, $e_2=\sqrt{(a^2+c^2)}$, $e_3=\sqrt{(b^2+c^2)}$, $x'=a\sec\theta\sec\varphi$; an element of the pyramid ABCFG at P is $x^2\cos\varphi d\theta d\varphi dx$; the limits of θ are 0 and θ'; of φ, 0 and φ'; and of x, 0 and x'. Hence, since the whole number of points within the pyramid is $\frac{1}{6}abc$, we have

$$\Delta_1=\frac{3}{abc}\int_0^{\theta'}\int_0^{\varphi'}\int_0^{x'}x^3\cos\varphi d\theta d\varphi dx=\frac{3a^3}{4bc}\int_0^{\theta'}\int_0^{\varphi'}\sec^4\theta\sec^3\varphi d\theta d\varphi,$$

$$=\frac{3a}{8bc}\int_0^{\theta'}\left[b\cos\theta\sqrt{(a^2+b^2\cos^2\theta)}+a^2\log\left\{\frac{b\cos\theta+\sqrt{(a^2+b^2\cos^2\theta)}}{a}\right\}\right]\sec^4\theta d\theta,$$

$$=\frac{e}{4}-\frac{a^3}{4bc}\tan^{-1}\left(\frac{bc}{ae}\right)+\left(\frac{3a^2+c^2}{8b}\right)\log\left(\frac{b+e}{e_2}\right)+\left(\frac{3a^2+b^2}{8c}\right)\log\left(\frac{c+e}{e_1}\right).$$ By symmetry we have

$$\Delta_2=\frac{e}{4}-\frac{b^3}{4ac}\tan^{-1}\left(\frac{ac}{be}\right)+\left(\frac{3b^2+a^2}{8c}\right)\log\left(\frac{c+e}{e_1}\right)+\left(\frac{3b^2+c^2}{8a}\right)\log\left(\frac{a+e}{e_3}\right),$$

$$\Delta_3=\frac{e}{4}-\frac{c^3}{4ab}\tan^{-1}\left(\frac{ab}{ce}\right)+\left(\frac{3c^2+a^2}{8b}\right)\log\left(\frac{b+e}{e_2}\right)+\left(\frac{3c^2+b^2}{8a}\right)\log\left(\frac{a+e}{e_3}\right).$$

$$\therefore\Delta=\tfrac{1}{3}(\Delta_1+\Delta_2+\Delta_3)=\frac{e}{4}-\frac{a^3}{12bc}\tan^{-1}\left(\frac{bc}{ae}\right)-\frac{b^3}{12ac}\tan^{-1}\left(\frac{ac}{be}\right)-\frac{c^3}{12ab}\tan^{-1}\left(\frac{ab}{ce}\right)$$
$$+\frac{e_1^2}{6c}\log\left(\frac{c+e}{e_1}\right)+\frac{e_2^2}{6b}\log\left(\frac{b+e}{e_2}\right)+\frac{e_3^2}{6a}\log\left(\frac{a+e}{e_3}\right).$$

Cor. If $a=b=c$, $\Delta=\frac{1}{4}a[\sqrt{3}-\frac{1}{6}\pi+2\log(2+\sqrt{3})]$.

II.—Solution by the Proposer; and H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, Cabarrus County, North Carolina.

Let x, y, z be co-ordinates of any point P within the solid, the origin being at the corner A. Then $AP=\sqrt{(x^2+y^2+z^2)}$, and if $\Delta=$ the average distance required we have

$$\Delta=\frac{1}{abc}\int_0^a\int_0^b\int_0^c\sqrt{(x^2+y^2+z^2)}dxdydz,$$

$$=\frac{1}{2ab}\int_0^a\int_0^b\sqrt{(c^2+x^2+y^2)}dxdy+\frac{1}{2abc}\int_0^a\int_0^b(x^2+y^2)\log\left(\frac{c+\sqrt{(c^2+x^2+y^2)}}{\sqrt{(x^2+y^2)}}\right)dxdy,$$

$$=\frac{1}{3a}\int_0^a\sqrt{(x^2+b^2+c^2)}dx+\frac{1}{6ab}\int_0^a(3x^2+c^2)\log\left(\frac{b+\sqrt{(x^2+b^2+c^2)}}{\sqrt{(x^2+c^2)}}\right)dx$$
$$+\frac{1}{6ac}\int_0^a(3x^2+b^2)\log\left(\frac{c+\sqrt{(x^2+b^2+c^2)}}{\sqrt{(x^2+b^2)}}\right)dx-\frac{1}{3abc}\int_0^a x^3\sin^{-1}\left(\frac{bc}{\sqrt{(x^2+b^2)}\sqrt{(x^2+c^2)}}\right)dx,$$

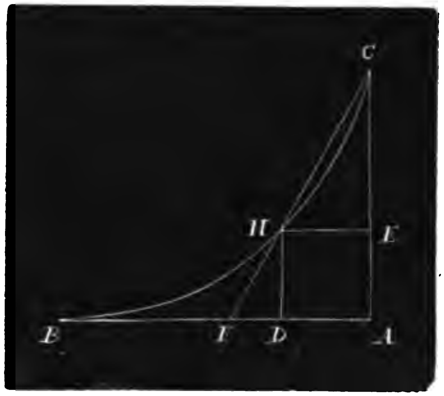
$$=\tfrac{1}{4}\sqrt{(a^2+b^2+c^2)}+\frac{(b^2+c^2)}{6a}\log\left(\frac{a+\sqrt{(a^2+b^2+c^2)}}{\sqrt{(b^2+c^2)}}\right)+\frac{(a^2+c^2)}{6b}\log\left(\frac{b+\sqrt{(a^2+b^2+c^2)}}{\sqrt{(a^2+c^2)}}\right)$$
$$+\frac{(a^2+b^2)}{6c}\log\left(\frac{c+\sqrt{(a^2+b^2+c^2)}}{\sqrt{(a^2+b^2)}}\right)-\frac{a^3}{12bc}\sin^{-1}\left(\frac{bc}{\sqrt{(a^2+b^2)}\sqrt{(a^2+c^2)}}\right)$$
$$-\frac{b^3}{12ac}\sin^{-1}\left(\frac{ac}{\sqrt{(a^2+b^2)}\sqrt{(b^2+c^2)}}\right)-\frac{c^3}{12ab}\sin^{-1}\left(\frac{ab}{\sqrt{(a^2+c^2)}\sqrt{(b^2+c^2)}}\right),$$

$$=\frac{e}{4}+\frac{(b^2+c^2)}{12a}\log\left(\frac{e+a}{e-a}\right)+\frac{(a^2+c^2)}{12b}\log\left(\frac{e+b}{e-b}\right)+\frac{(a^2+b^2)}{12c}\log\left(\frac{e+c}{e-c}\right)$$
$$-\frac{a^3}{12bc}\tan^{-1}\left(\frac{bc}{ae}\right)-\frac{b^3}{12ac}\tan^{-1}\left(\frac{ac}{be}\right)-\frac{c^3}{12ab}\tan^{-1}\left(\frac{ab}{ce}\right).$$

84.—Proposed by E. F. Norton, Allen, Hillsdale County, Michigan.

A fox starts from a point 100 rods due north from a hound, and runs due west at the rate of 10 miles an hour; at the same instant the hound starts in pursuit, at the rate of 15 miles an hour, always keeping in a direct line between his starting point and the fox. Required the equation of the *curve* described by the hound, and the distance he runs to catch the fox.—[From the *Normal Monthly*, vol. iii, p. 85.

I.—Solution by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.



Let A be the starting point of the fox, C the starting point of the hound, B the point where the fox is caught, and the curve CB the path of the hound. Take any point H in the curve, and from C draw a line through H meeting AB in F; then when the hound is at H the fox will be at F. Draw HE parallel, and HD perpendicular to AB. Take the origin of rectangular co-ordinates at C, and put AC $= a$, $n = \frac{15}{10} =$ ratio of speed, EH $=$ AD $= x$, CE $= y$, AF $= w$ and curve CH $= z$. From the similar triangles HCE, FCA we have $x : y :: w : a$, whence $w = \frac{ax}{y}$........................(1).

Also, $z = nw = \frac{nax}{y}$.......(2), and $z = \int\sqrt{(dx^2+dy^2)}$.......(3).

$\therefore \quad \int\sqrt{(dx^2+dy^2)} = \frac{nax}{y}$...(4), whence, differentiating,

$\sqrt{(dx^2+dy^2)} = \frac{na(ydx - xdy)}{y^2}$..............(5). Putting $\frac{dx}{dy} = p$,

(5) becomes $y^2\sqrt{(1+p^2)} = napy - nax$...........................(6).

Differentiating (6) we get $\frac{ypdp}{\sqrt{(1+p^2)}} + 2dy\sqrt{(1+p^2)} = nadp$..(7).

Dividing by $2(1+p^2)^{\frac{1}{2}}$, $dy(1+p^2)^{\frac{1}{2}} + \frac{ypdp}{2(1+p^2)^{\frac{1}{2}}} = \frac{nadp}{2(1+p^2)^{\frac{1}{2}}}$...(8). Integrating (8), $y(1+p^2)^{\frac{1}{2}} = \frac{na}{2}\int\frac{dp}{(1+p^2)^{\frac{1}{2}}}$..(9).

Let $p = 2w\sqrt{(w^2-1)}$, then $dp = \frac{2(2w^2-1)dw}{\sqrt{(w^2-1)}}$, and $y(1+p^2)^{\frac{1}{2}} = na\int\frac{dw\sqrt{(2w^2-1)}}{\sqrt{(w^2-1)}}$........................(10).

$\therefore \quad y(1+p^2)^{\frac{1}{2}} = na\text{H}(\sqrt{2}, w) + \text{C}$, where the notation $\text{H}(\sqrt{2}, w)$ denotes an arc of an equilateral hyperbola, semi-axes 1 and abscissa w. When $y = 0$, $x = 0$, $p = 0$ and $w = 1$; therefore $\text{C} = 0$, and

$$y(1+p^2)^{\frac{1}{2}} = na\text{H}(\sqrt{2}, w)\text{...(11).}$$

But $w = \sqrt{[\frac{1}{2} + \frac{1}{2}\sqrt{(p^2+1)}]}$ and from (6) $p = \frac{n^2a^2x \pm y\sqrt{[n^2a^2(x^2+y^2)-y^4]}}{y(n^2a^2-y^2)}$,

$$\therefore y\left[1+\left(\frac{n^2a^2x+y\sqrt{[n^2a^2(x^2+y^2)-y^4]}}{y(n^2a^2-y^2)}\right)^2\right]^{\frac{1}{2}} = na\text{H}\left(\sqrt{2}, \sqrt{\frac{1}{2}+\frac{1}{2}\left[1+\left(\frac{n^2a^2x+y\sqrt{[n^2a^2(x^2+y^2)-y^4]}}{y(n^2a^2-y^2)}\right)^2\right]^{\frac{1}{2}}}\right)$$

is the equation to the curve described by the hound.

When the hound overtakes the fox, $y = a$ and (6) gives $x = ap - \frac{a\sqrt{(1+p^2)}}{n}$.

We can integrate (9) by series, and then by reversion find p in terms of y; and then when $y = a$ we have p in terms of n; but the resulting series converges slowly when n is less than 2.

II.—Solution by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Let CE $= y$, HE $= x$, AF $= w$, CH $= z$, CA $= a$, $n = \frac{15}{10} = \frac{3}{2}$. $a : w :: y : x$, and $w : z :: 1 : n$; $\therefore \; ax = wy$...(1) and $z = nw$...(2). Differentiating (1) and (2), $adx = wdy + ydw$...(3), and $dz = ndw$...(4). Assume $dx = \sin\varphi dz$, then $dy = \cos\varphi dz$ and by substitution we get from (3), $y = na\sin\varphi - nw\cos\varphi$...(5). Differentiating (5), $dy = na\cos\varphi d\varphi + nw\sin\varphi d\varphi - n\cos\varphi dw$. But $dy = \cos\varphi dz = n\cos\varphi dw$; substituting we get $2\cos\varphi dw - w\sin\varphi d\varphi = a\cos\varphi d\varphi$...(6). Dividing by $\sqrt{(\cos\varphi)}$,

$2dw\sqrt{(\cos\varphi)} - \frac{w\sin\varphi d\varphi}{\sqrt{(\cos\varphi)}} = ad\varphi\sqrt{(\cos\varphi)}$.........(7). Integrating, $2w\sqrt{(\cos\varphi)} = a\int d\varphi\sqrt{(\cos\varphi)}$.........(8).

Let $\cos\varphi = s^2$, then $\varphi = \cos^{-1}s^2$, $d\varphi = -\frac{2sds}{\sqrt{(1-s^4)}}$, and $ws = -a\int\frac{s^2ds}{\sqrt{(1-s^2)}\sqrt{(1+s^2)}}$................(9).

Let $1 - s^2 = t^2$, then $s = \sqrt{(1-t^2)}$, $ds = -\frac{tdt}{\sqrt{(1-t^2)}}$, and $ws = a\int\frac{\sqrt{(1-t^2)}dt}{\sqrt{(2-t^2)}}$,

$$= a\sqrt{2}\int\frac{\sqrt{(1-\frac{1}{2}t^2)}dt}{\sqrt{(1-t^2)}} - \frac{1}{2}a\sqrt{2}\int\frac{dt}{\sqrt{(1-t^2)}\sqrt{(1-\frac{1}{2}t^2)}} = a\sqrt{2}[\text{E}(\tfrac{1}{2}\sqrt{2}, t) - \tfrac{1}{2}\text{F}(\tfrac{1}{2}\sqrt{2}, t)] + C.$$

When $y=0$, $x=0$, $w=0$, $\varphi=0$, $s=1$ and $t=0$; $\therefore$ $C=0$, $w=a\sqrt{2}[\mathrm{E}(\frac{1}{2}\sqrt{2},\ t)-\frac{1}{2}\mathrm{F}(\frac{1}{2}\sqrt{2},\ t)]$,

$$=a\sqrt{2}[\mathrm{E}(\tfrac{1}{2}\sqrt{2},\ \sqrt{1-\cos\varphi})-\tfrac{1}{2}\mathrm{F}(\tfrac{1}{2}\sqrt{2},\ \sqrt{1-\cos\varphi})]\quad\text{.................................(10).}$$

From (1), $w=\frac{ax}{y}$; substituting in (5), $y^2=nay\sin\varphi-nax\cos\varphi$. Solving for $\cos\varphi$ we find

$$\cos\varphi=\frac{y\sqrt{[n^2a^2(x^2+y^2)-y^4]}-xy^2}{na(x^2+y^2)}.\quad\text{Substituting in (10), } \frac{x}{y}\left(\frac{y\sqrt{[n^2a^2(x^2+y^2)-y^4]}-xy^2}{na(x^2+y^2)}\right)^{\frac{1}{2}}$$

$$=\sqrt{2}\mathrm{E}\left(\tfrac{1}{2}\sqrt{2},\ \sqrt{1-\frac{y\sqrt{[n^2a^2(x^2+y^2)-y^4]}-xy^2}{na(x^2+y^2)}}\right)-\tfrac{1}{2}\sqrt{2}\,\mathrm{F}\left(\tfrac{1}{2}\sqrt{2},\ \sqrt{1-\frac{y\sqrt{[n^2a^2(x^2+y^2)-y^4]}-xy^2}{na(x^2+y^2)}}\right),$$

which is the rectangular equation to the curve described by the hound.

I am indebted to *Leybourn's Mathematical Repository*, New Series, Vol. 3, pp. 205—6, for the methods of integration employed in this and the preceding solution.

85.—Proposed by J. J. Sylvester, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

Let X be the coefficient of any power of a in the expansion of $\frac{1}{(1-xa)(1-x^2a)(1-x^3a)\ldots(1-x^ia)}$, where i is any integer, and let X be arranged according to the powers of x; prove that its coefficient will form a series reading the same from left to right as from right to left, possessing this property that as we pass from either end toward the central term or terms the coefficients may increase or remain unaltered but can never decrease.

Solution by Rev. W. J. Wright, M. A., Ph. D., Member of the London Mathematical Society, Cape May Point, Cape May Co., N. J.

Let $f(x_0+x_1a+x_2a^2+\ldots)=X_0+X_1a+X_2a^2+\ldots$(1) in which is represented the expanded denominator of the given fraction.

By the Theorem of Arbogast any term of this expansion, as $X_m=D^{m-1}x_1\,.\,f_1x_0+D^{m-2}x_1^2\,.\,f_2x_0+\ldots\ldots+Dx_1^{m-1}\,.\,f_{m-1}x_0+x_1^mf_mx_0$; in which D, D^2,.......D^{m-1} are derivatives obtained by differentiating (1) as a function of two variables, and which are shown to follow in an invariable order with coefficients which are obtained by *divided* differentiation, in which also by a similar process of differentiation the derivatives of fx_0 are expanded into a series, as $D^mfx_0=D^{m-1}x_1\,.\,f_1x_0+D^{m-2}x_1^2\,.\,f_2x_0+\ldots\ldots+x_1^m\,.\,f_mx_0$.

In this case we have to expand $(1-x_1a+x_2a^2-\ldots\ldots)^{-1}$ in which we know the value of x_1 to be $x+x^2+\ldots\ldots+x^i$ and $x_0=1$; hence $X_0=1$ and $X_1=x+x^2+\ldots\ldots+x^i$.

The equation above for the values of X_2, ... X_i may readily be found when the derivatives have been written out as $Dx_1=x_2$, $D^2x_1=x_3$; $Dx_1^2=2x_1x_2$, $D^2x^2_1=2x_1x_3+x_2^2$. This indeed is a little different from the derivation of Arbogast, but it immediately follows from a divided differentiation of (1). The derivatives of $fx_0=f1$ are seen to be $D^mf1=-D^{m-1}x_1+D^{m-2}x_1^2-\ldots\ldots\pm x_1^m$, and the expansion is $(1-x_1a+x_2a^2-\ldots\ldots)^{-1}=1+X_1+(x_1^2-Dx_1)a^2+(x_1^3-Dx_1^2+D^2x_1)a^3+\ldots\ldots$

Now it will be observed that in this expansion x_1 is the leading letter in each term, and since it is the sum of powers its second and other powers as $(x+x^2+\ldots\ldots+x^i)^n$ will give coefficients which increase to the central term or terms and can not decrease, a relation which manifestly can not be changed by any of the succeeding terms in X_n.

Illustration. — Let $i=3$, then $X_3=(x+x^2+x^3)^3-2x_1x_2+x_3=x^3+x^4+2x^5+2x^6+2x^7+x^8+x^9$, where $2x_1x_2=2(x+x^2+x^3)(x^3+x^4+x^5)$ and $x_3=x^6$.

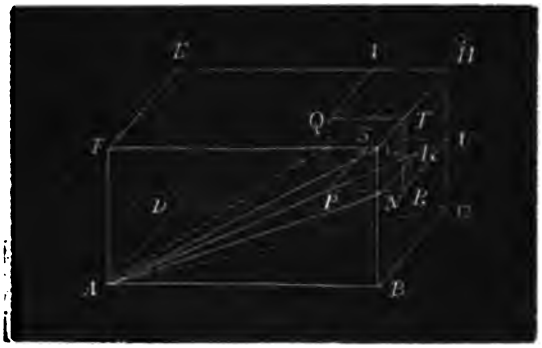
Professor Sylvester sent the following note with the problem:

"The questioner proposes to call such series *graded* series. It may be *proved* that the multiplier of every combination of powers of a, b, c, ... in the expansion of $\frac{1}{(a,\ x,\ i)(b,\ x,\ j)(c,\ x,\ k)\ldots}$ where a, x, i represents $(1-a)(1-xa)(1-x^2a)\ldots(1-x^ia)$ will also form a graded series. It *seems* also to be the case that the expansion of $(a\ \ x,\ i)(b,\ x.\ j)(c,\ x,\ k)\ldots$ possesses the same property of generating graded series, and that a very great further generalization may be made of the theorem by introducing powers, and also by changing the form of the simple factors, the function $(a,\ x,\ i)$ always of course retaining the general form $\varphi1\ .\ \varphi x\ .\ \varphi x^2\ .\ \varphi x^3\ldots\varphi x^i$; but beyond the theorem first stated in this note the proposer of the question has not proceeded in the way of proof, nor does his method of proof for this case appear capable of being extended beyond it."

86.—Proposed by E. B. Seitz, Greenville, Darke County, Ohio.

Find the average distance between two points taken at random within a rectangular solid, edges a, b, c.

I.—Solution by the Proposer.



Let AH be the rectangular solid. Draw AH, the diagonal of the solid, and AC, AG, AE, the diagonals of the faces. The solid consists of the three pyramids, ABCHG, ACDEH, AEFGH.

From A draw AP to any point P within the pyramid ABCHG, produce it to K in the face BCHG, and draw KN perpendicular to BG. Form the rectangular solid PRUSVHTQ.

Now if AP is the distance and the direction of the second random point from the first, the volume of the solid PH represents the number of ways the two points can be taken.

Let $AB = a$, $AD = b$, $AF = c$, $AH = e = (a^2+b^2+c^2)^{\frac{1}{2}}$, $AC = e_1 = (a^2+b^2)^{\frac{1}{2}}$, $AG = e_2 = (a^2+c^2)^{\frac{1}{2}}$, $AE = e_3 = (b^2+c^2)^{\frac{1}{2}}$, $AP = x$, $AK = x'$, $\angle NAB = \theta$, $\angle PAN = \varphi$, $\theta' = \tan^{-1}\left(\frac{c}{a}\right)$, $\varphi' = \tan^{-1}\left(\frac{b\cos\theta}{a}\right)$; and let Δ = the required average, Δ_1, Δ_2, Δ_3 = the average distances between the two points when P is within the pyramids ABCHG, ACDEH, AEFGH, respectively.

Then $x' = a\sec\theta\sec\varphi$, volume $PH = (a - x\cos\theta\cos\varphi)(b - x\sin\varphi)(c - x\sin\theta\cos\varphi)$; an element of the solid at P is $x^2\cos\varphi d\theta d\varphi dx$; for Δ_1, the limits of θ are 0 and θ'; of φ, 0 and φ'; and of x, 0 and x'.

$$\therefore \Delta_1 = \frac{\int_0^{\theta'}\int_0^{\phi'}\int_0^{x'}(a - x\cos\theta\cos\varphi)(b - x\sin\varphi)(c - x\sin\theta\cos\varphi)x^3\cos\varphi d\theta d\varphi dx}{\int_0^{\theta'}\int_0^{\phi'}\int_0^{x'}(a - x\cos\theta\cos\varphi)(b - x\sin\varphi)(c - x\sin\theta\cos\varphi)x^2\cos\varphi d\theta d\varphi dx},$$

$$= \frac{24}{a^2b^2c^2}\int_0^{\theta'}\int_0^{\phi'}\int_0^{x'}(a - x\cos\theta\cos\varphi)(b - x\sin\varphi)(c - x\sin\theta\cos\varphi)x^3\cos\varphi d\theta d\varphi dx,$$

$$= \frac{2a^3}{35b^2c^2}\int_0^{\theta'}\int_0^{\phi'}(21bc - 14ab\tan\theta - 14ac\sec\theta\tan\varphi + 10a^2\sin\theta\sec^2\theta\tan\varphi)\sec^4\theta\sec^3\varphi d\theta d\varphi,$$

$$= \frac{a}{105b^2c^2}\int_0^{\theta'}\Big[28a^3c\sec^5\theta - 20a^4\sin\theta\sec^6\theta + 63b^3c\sec^3\theta(a^2+b^2\cos^2\theta)^{\frac{1}{2}} - 42ab^2\sin\theta\sec^4\theta(a^2+b^2\cos^2\theta)^{\frac{1}{2}}$$

$$- 28c\sec^5\theta(a^2+b^2\cos^2\theta)^{\frac{3}{2}} + 20a\sin\theta\sec^6\theta(a^2+b^2\cos^2\theta)^{\frac{3}{2}} + 63a^2bc\sec^4\theta\log\left(\frac{b\cos\theta + (a^2+b^2\cos^2\theta)^{\frac{1}{2}}}{a}\right)$$

$$- 42a^3b\sin\theta\sec^5\theta\log\left(\frac{b\cos\theta + (a^2+b^2\cos^2\theta)^{\frac{1}{2}}}{a}\right)\Big]d\theta,$$

$$= \frac{e}{5} + \frac{4a^5 + 4e^5 - 4e_1^5 - 4e_2^5}{105b^2c^2} + \frac{a^2e_1}{6c^2} + \frac{a^2e_2}{6b^2} + \frac{b^2e_1}{15c^2} + \frac{c^2e_2}{15b^2} - \frac{e^3e_3^2}{15b^2c^2} - \frac{a^2ee_3^2}{10b^2c^2} - \frac{2a^3}{5bc}\tan^{-1}\left(\frac{bc}{ae}\right)$$

$$+ \frac{a^4}{10bc^2}\log\left(\frac{b+e_1}{a}\right) + \frac{a^4}{10b^2c}\log\left(\frac{c+e_2}{a}\right) + \left(\frac{2a^2}{5b} + \frac{c^2}{10b} - \frac{a^4}{10bc^2}\right)\log\left(\frac{b+e}{e_2}\right) + \left(\frac{2a^2}{5c} + \frac{b^2}{10c} - \frac{a^4}{10b^2c}\right)\log\left(\frac{c+e}{e_1}\right).$$

The values of Δ_2 and Δ_3 are similar to that of Δ_1, and may be written out without performing the integrations.

$$\therefore \Delta = \tfrac{1}{3}(\Delta_1 + \Delta_2 + \Delta_3) = \frac{e}{15} + \frac{4(a^7 + b^7 + c^7 + e^7 - e_1^7 - e_2^7 - e_3^7)}{315a^2b^2c^2} + \frac{7}{90}\left(\frac{e_1^3}{c^2} + \frac{e_2^3}{b^2} + \frac{e_3^3}{a^2}\right)$$

$$- \frac{7e}{90}\left(\frac{e_1^2}{c^2} + \frac{e_2^2}{b^2} + \frac{e_3^2}{a^2}\right) - \frac{2a^3}{15bc}\tan^{-1}\left(\frac{bc}{ae}\right) - \frac{2b^3}{15ac}\tan^{-1}\left(\frac{ac}{be}\right) - \frac{2c^3}{15ab}\tan^{-1}\left(\frac{ab}{ce}\right)$$

$$+ \frac{a^4}{30b^2c^2}\left[b\log\left(\frac{b+e_1}{a}\right) + c\log\left(\frac{c+e_2}{a}\right)\right] + \frac{b^4}{30a^2c^2}\left[a\log\left(\frac{a+e_1}{b}\right) + c\log\left(\frac{c+e_3}{b}\right)\right]$$

$$+ \frac{c^4}{20a^2b^2}\left[a\log\left(\frac{a+e_2}{c}\right) + b\log\left(\frac{b+e_3}{c}\right)\right] + \tfrac{1}{6}\left(\frac{a^2+b^2}{c} - \frac{a^6+b^6}{5a^2b^2c}\right)\log\left(\frac{c+e}{e_1}\right)$$

$$+ \tfrac{1}{6}\left(\frac{a^2+c^2}{b} - \frac{a^6+c^6}{5a^2bc^2}\right)\log\left(\frac{b+e}{e_2}\right) + \tfrac{1}{6}\left(\frac{b^2+c^2}{a} - \frac{b^6+c^6}{5ab^2c^2}\right)\log\left(\frac{a+e}{e_3}\right).$$

Cor.—If $a = b = c$, $\Delta = \tfrac{1}{5}a\left(\frac{4}{21} + \frac{17\sqrt{2}}{21} - \frac{2\sqrt{3}}{7} - \frac{\pi}{3} + \log[(1+\sqrt{2})(7+4\sqrt{3})]\right).$

II.—Solution by H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, Cabarrus County, North Carolina; and Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Taking one corner of the solid as origin of rectangular co-ordinates, denote the points by (x, y, z) and (w, v, s) respectively, and the required average distance by $\varDelta$; then the distance between the points is $[(x-w)^2+(y-v)^2+(z-s)^2]^{\frac12}$, and

$$\varDelta=\frac{\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z[(x-w)^2+(y-v)^2+(z-s)^2]^{\frac12}dxdydzdwdvds}{\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z dxdydzdwdvds},$$

$$=\frac{8}{a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z[(x-w)^2+(y-v)^2+(z-s)^2]^{\frac12}dxdydzdwdvds,$$

$$=\frac{4}{a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\Big[z[(x-w)^2+(y-v)^2+z^2]^{\frac12}$$
$$+[(x-w)^2+(y-v)^2]\log\Big(\frac{z+[(x-w)^2+(y-v)^2+z^2]^{\frac12}}{[(x-w)^2+(y-v)^2]^{\frac12}}\Big)\Big]dxdydzdwdv,$$

$$=\frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\Big[2yz[(x-w)^2+y^2+z^2]^{\frac12}+[3z(x-w)^2+z^3]\log\Big(\frac{y+[(x-w)^2+y^2+z^2]^{\frac12}}{[(x-w)^2+z^2]^{\frac12}}\Big)$$
$$+[3y(x-w)^2+y^3]\log\Big(\frac{z+[(x-w)^2+y^2+z^2]^{\frac12}}{[(x-w)^2+y^2]^{\frac12}}\Big)-2(x-w)^3\tan^{-1}\Big(\frac{yz}{(x-w)[(x-w)^2+y^2+z^2]^{\frac12}}\Big)\Big]dxdydzdw,$$

$$=\frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\int_0^c\Big[\frac{3xyz(x^2+y^2+z^2)^{\frac12}}{2}+xy(x^2+y^2)\log\Big(\frac{z+(x^2+y^2+z^2)^{\frac12}}{(x^2+y^2)^{\frac12}}\Big)+xz(x^2+z^2)\log\Big(\frac{y+(x^2+y^2+z^2)^{\frac12}}{(x^2+z^2)^{\frac12}}\Big)$$
$$+yz(y^2+z^2)\log\Big(\frac{x+(x^2+y^2+z^2)^{\frac12}}{(y^2+z^2)^{\frac12}}\Big)-\frac{x^4}{2}\tan^{-1}\Big(\frac{yz}{x(x^2+y^2+z^2)^{\frac12}}\Big)-\frac{y^4}{2}\tan^{-1}\Big(\frac{xz}{y(x^2+y^2+z^2)^{\frac12}}\Big)$$
$$-\frac{z^4}{2}\tan^{-1}\Big(\frac{xy}{z(x^2+y^2+z^2)^{\frac12}}\Big)\Big]dxdydz,$$

$$=\frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\Big[\frac{7xy(x^2+y^2)^{\frac32}}{20}-\frac{7xy(x^2+y^2+c^2)^{\frac32}}{20}+\frac{19c^2xy(x^2+y^2+c^2)^{\frac12}}{20}+cxy(x^2+y^2)\log\Big(\frac{c+(x^2+y^2+c^2)^{\frac12}}{(x^2+y^2)^{\frac12}}\Big)$$
$$+\frac{8c^2x(2x^2+c^2)-3x(x^2+c^2)^2}{20}\log\Big(\frac{y+(x^2+y^2+c^2)^{\frac12}}{(x^2+c^2)^{\frac12}}\Big)+\frac{8c^2y(2y^2+c^2)-3y(y^2+c^2)^2}{20}\log\Big(\frac{x+(x^2+y^2+c^2)^{\frac12}}{(y^2+c^2)^{\frac12}}\Big)$$
$$+\frac{8x^5}{20}\log\Big(\frac{y+(x^2+y^2)^{\frac12}}{x}\Big)+\frac{3y^5}{20}\log\Big(\frac{x+(x^2+y^2)^{\frac12}}{y}\Big)-\frac{cx^4}{2}\tan^{-1}\Big(\frac{cy}{x(x^2+y^2+c^2)^{\frac12}}\Big)-\frac{cy^4}{2}\tan^{-1}\Big(\frac{cx}{y(x^2+y^2+c^2)^{\frac12}}\Big)$$
$$-\frac{c^5}{10}\tan^{-1}\Big[\frac{xy}{c(x^2+y^2+c^2)^{\frac12}}\Big]\Big]dxdy,=\frac{4}{3a^2b^2c^2}\int_0^a\Big[\frac{x^6}{15}-\frac{x(x^2+b^2)^{\frac52}}{15}+\frac{4b^2x(x^2+b^2)^{\frac32}}{15}-\frac{b^4x(x^2+b^2)^{\frac12}}{8}$$
$$-\frac{x(x^2+c^2)^{\frac52}}{15}+\frac{4c^2x(x^2+c^2)^{\frac32}}{15}-\frac{c^4x(x^2+c^2)^{\frac12}}{8}+\frac{x(x^2+b^2+c^2)^{\frac52}}{15}-\frac{4x(b^2+c^2)(x^2+b^2+c^2)^{\frac32}}{15}$$
$$+\frac{x(b^2+c^2)^2(x^2+b^2+c^2)^{\frac12}}{8}+\frac{2b^2c^2x(x^2+b^2+c^2)^{\frac12}}{5}+\frac{8b^2cx(2x^2+b^2)-3cx(x^2+b^2)^2}{20}\log\Big[\frac{c+(x^2+b^2+c^2)^{\frac12}}{(x^2+b^2)^{\frac12}}\Big]$$
$$+\frac{8bc^2x(2x^2+c^2)-3bx(x^2+c^2)^2}{20}\log\Big[\frac{b+(x^2+b^2+c^2)^{\frac12}}{(x^2+c^2)^{\frac12}}\Big]-\frac{(b^2+c^2)(b^4-6b^2c^2+c^4)}{40}\log\Big[\frac{x+(x^2+b^2+c^2)^{\frac12}}{(b^2+c^2)^{\frac12}}\Big]$$
$$+\frac{3bx^5}{20}\log\Big[\frac{b+(x^2+b^2)^{\frac12}}{x}\Big]+\frac{b^6}{40}\log\Big[\frac{x+(x^2+b^2)^{\frac12}}{b}\Big]+\frac{3cx^5}{20}\log\Big[\frac{c+(x^2+c^2)^{\frac12}}{x}\Big]+\frac{c^6}{40}\log\Big[\frac{x+(x^2+c^2)^{\frac12}}{c}\Big]$$
$$-\frac{cx^4}{2}\tan^{-1}\Big[\frac{bc}{x(x^2+b^2+c^2)^{\frac12}}\Big]-\frac{b^5c}{10}\tan^{-1}\Big[\frac{cx}{b(x^2+b^2+c^2)^{\frac12}}\Big]-\frac{bc^5}{10}\tan^{-1}\Big[\frac{bx}{c(x^2+b^2+c^2)^{\frac12}}\Big]\Big]dx,$$

$$=\frac{e}{15}+\frac{4[a^7+b^7+c^7+e^7-(a^2+b^2)^{\frac72}-(a^2+c^2)^{\frac72}-(b^2+c^2)^{\frac72}]}{315a^2b^2c^2}+\frac{7}{90}\Big[\frac{(a^2+b^2)^{\frac52}}{c^2}+\frac{(a^2+c^2)^{\frac52}}{b^2}+\frac{(b^2+c^2)^{\frac52}}{a^2}\Big]$$
$$-\frac{7e}{90}\Big[\frac{a^4+b^4}{a^2b^2}+\frac{a^4+c^4}{a^2c^2}+\frac{b^4+c^4}{b^2c^2}\Big]+\frac{1}{12}\Big[\frac{a^2+b^2}{c}-\frac{a^6+b^6}{5a^2b^2c}\Big]\log\Big[\frac{e+c}{e-c}\Big]+\frac{1}{12}\Big[\frac{a^2+c^2}{b}-\frac{a^6+c^6}{5a^2bc^2}\Big]\log\Big[\frac{e+b}{e-b}\Big]$$
$$+\frac{1}{12}\Big[\frac{b^2+c^2}{a}-\frac{b^6+c^6}{5ab^2c^2}\Big]\log\Big[\frac{e+a}{e-a}\Big]+\frac{a^4}{30bc^2}\log\Big[\frac{b+(a^2+b^2)^{\frac12}}{a}\Big]+\frac{a^4}{30b^2c}\log\Big[\frac{c+(a^2+c^2)^{\frac12}}{a}\Big]$$
$$+\frac{b^4}{30ac^2}\log\Big[\frac{a+(a^2+b^2)^{\frac12}}{b}\Big]+\frac{b^4}{30a^2c}\log\Big[\frac{c+(b^2+c^2)^{\frac12}}{b}\Big]+\frac{c^4}{30ab^2}\log\Big[\frac{a+(a^2+c^2)^{\frac12}}{c}\Big]$$
$$+\frac{c^4}{30a^2b}\log\Big[\frac{b+(b^2+c^2)^{\frac12}}{c}\Big]-\frac{2a^3}{15bc}\tan^{-1}\Big[\frac{bc}{ae}\Big]-\frac{2b^3}{15ac}\tan^{-1}\Big[\frac{ac}{be}\Big]-\frac{2c^3}{15ab}\tan^{-1}\Big[\frac{ab}{ce}\Big].$$

87.—Prize Problem. Proposed by Artemas Martin, M. A., Erie, Erie County, Pennsylvania.

A cask containing a gallons of wine is placed upon another containing b gallons of brandy. Water runs in at the top of the wine cask at the rate of m gallons per minute, the mixture escapes into the brandy cask at the same rate, the mixture in the brandy cask escapes at a like rate into a tub containing c gallons of water and the tub overflows. Find the quantity of each fluid in the tub at the end of t minutes, supposing them to mingle perfectly.

Solution by William Woolsey Johnson, Professor of Mathematics, St. John's College, Annapolis, Anne Arundel Co., Md.

1. Let x denote the number of gallons of wine in the first cask at the end of t minutes, y the number of gallons in the second cask, and z the number of gallons in the tub.

The rate per minute at which wine is passing from the first into the second cask is $\frac{mx}{a}$; this fraction also represents the rate at which wine is entering the second cask, while $\frac{my}{b}$ is the rate at which wine is leaving it. Thus we have the differential equations:

$$\frac{dx}{dt}=-\frac{mx}{a}\text{.......(1)},\quad \frac{dy}{dt}=\frac{mx}{a}-\frac{my}{b}\text{.......(2)},\quad \frac{dz}{dt}=\frac{my}{b}-\frac{mz}{c}\text{.......(3)};$$

and to determine the constants of integration we have, when $t=0$, $x=a$, $y=0$, $z=0$. From (1), $\frac{dx}{x}=-\frac{mdt}{a}$; $\therefore\ x=e^{-\frac{mt}{a}}+$ constant, and, since $t=0$ gives $x=a$, $x=ae^{-\frac{mt}{a}}$......(4). Substituting this value of x, (2) becomes $\frac{dy}{dt}=me^{-\frac{mt}{a}}-\frac{my}{b}$........(5).

Since the solution would be of the form $y=Be^{-\frac{mt}{b}}$...(6) were the first term of (5) zero, we may assume the solution to be of the form (6), B denoting a function of t.

Differentiating (6), $\frac{dy}{dt}=e^{-\frac{mt}{b}}\left(\frac{dB}{dt}\right)-\frac{m}{b}Be^{-\frac{mt}{b}}$, which agrees with (5) if $\frac{dB}{dt}=me^{\frac{mt}{b}-\frac{mt}{a}}$...............(7); from which $B=\frac{ab}{a-b}e^{\frac{mt}{b}-\frac{mt}{a}}+k$; but, since $t=0$ gives $y=0$, and therefore, by (6), $B=0$, we have $k=-\frac{ab}{a-b}$, and substituting the value of B and k in (6) $y=\frac{ab}{a-b}\left(e^{-\frac{mt}{a}}-e^{-\frac{mt}{b}}\right)$............................(8).

Substituting this value of y, (3) becomes $\frac{dz}{dt}=\frac{ma}{a-b}\left(e^{-\frac{mt}{a}}-e^{-\frac{mt}{b}}\right)-\frac{mz}{c}$.......(9), whence, as above, we assume $z=Be^{-\frac{mt}{c}}$.......(10); differentiating, $\frac{dz}{dt}=e^{-\frac{mt}{c}}\frac{dB}{dt}-\frac{m}{c}Be^{-\frac{mt}{c}}$, which agrees with (9) if

$$\frac{dB}{dt}=\frac{ma}{a-b}\left(e^{\frac{mt}{c}-\frac{mt}{a}}-e^{\frac{mt}{c}-\frac{mt}{b}}\right).$$

Integrating, $B=\frac{a^2c}{(a-b)(a-c)}e^{\frac{mt}{a}-\frac{mt}{b}}-\frac{abc}{(a-b)(b-c)}e^{\frac{mt}{c}-\frac{mt}{b}}+k$; but since $t=0$ gives $z=0$, and therefore $B=0$, we have $k=\frac{ac^2}{(a-c)(b-c)}$.

Substituting the value of B and k in (10),

$$z=ac\left(\frac{a}{(a-b)(a-c)}e^{-\frac{mt}{a}}-\frac{b}{(a-b)(b-c)}e^{-\frac{mt}{b}}+\frac{c}{(a-c)(b-c)}e^{-\frac{mt}{c}}\right)\text{.................. (11)}.$$

This gives the wine in the tub; the brandy in the tub is found by the proper change of constants in (7) to be $\frac{bc}{b-c}\left(e^{-\frac{mt}{b}}-e^{-\frac{mt}{c}}\right)$; whence the water in the tub is

$$c-\frac{a^2c}{(a-b)(a-c)}e^{-\frac{mt}{a}}+\frac{b^2c}{(a-b)(b-c)}e^{-\frac{mt}{b}}-\frac{c(ac+bc-ab)}{(a-c)(b-c)}e^{-\frac{mt}{c}}.$$

2. The solution takes a different form when two or more of the constants a, b, c are equal. Thus if $b=a$, we have in place of (7), $\frac{dB}{dt}=m$; whence $B=mt+k$, in which $t=0$ gives $k=0$; therefore $y=mte^{-\frac{mt}{a}}$......(12). We have now, in place of equation (9,) $\frac{dz}{dt}=\frac{m^2t}{a}e^{-\frac{mt}{a}}-\frac{mz}{c}$; and assuming as before $z=Be^{-\frac{mt}{c}}$, $\frac{dB}{dt}=\frac{m^2t}{a}e^{\frac{mt}{c}-\frac{mt}{a}}$....(13), whence, integrating, $B=\frac{mct}{a-c}e^{\frac{mt}{c}-\frac{mt}{a}}-\frac{ac^2}{(a-c)^2}e^{\frac{mt}{c}-\frac{mt}{a}}+k$ in which $t=0$ gives $k=\frac{ac^2}{(a-c)^2}$; hence substituting we have $z=\frac{mct}{a-c}e^{-\frac{mt}{a}}-\frac{ac^2}{(a-c)^2}e^{-\frac{mt}{a}}+\frac{ac^2}{(a-c)^2}e^{-\frac{mt}{c}}$...(14).

II.—Solution by H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, Cabarrus County, North Carolina; and Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Taking one corner of the solid as origin of rectangular co-ordinates, denote the points by (x, y, z) and (w, v, s) respectively, and the required average distance by Δ; then the distance between the points is $[(x-w)^2+(y-v)^2+(z-s)^2]^{\frac{1}{2}}$, and

$$\Delta = \frac{\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z [(x-w)^2+(y-v)^2+(z-s)^2]^{\frac{1}{2}}dxdydzdwdvds}{\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z dxdydzdwdvds},$$

$$= \frac{8}{a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\int_0^z [(x-w)^2+(y-v)^2+(z-s)^2]^{\frac{1}{2}}dxdydzdwdvds,$$

$$= \frac{4}{a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\int_0^y\Big[z[(x-w)^2+(y-v)^2+z^2]^{\frac{1}{2}} + [(x-w)^2+(y-v)^2]\log\Big(\frac{z+[(x-w)^2+(y-v)^2+z^2]^{\frac{1}{2}}}{[(x-w)^2+(y-v)^2]^{\frac{1}{2}}}\Big)\Big]dxdydzdwdv,$$

$$= \frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\int_0^c\int_0^x\Big[2yz[(x-w)^2+y^2+z^2]^{\frac{1}{2}} + [3z(x-w)^2+z^3]\log\Big(\frac{y+[(x-w)^2+y^2+z^2]^{\frac{1}{2}}}{[(x-w)^2+z^2]^{\frac{1}{2}}}\Big)$$
$$+[3y(x-w)^2+y^3]\log\Big(\frac{z+[(x-w)^2+y^2+z^2]^{\frac{1}{2}}}{[(x-w)^2+y^2]^{\frac{1}{2}}}\Big) - 2(x-w)^3\tan^{-1}\Big(\frac{yz}{(x-w)[(x-w)^2+y^2+z^2]^{\frac{1}{2}}}\Big)\Big]dxdydzdw,$$

$$= \frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\int_0^c\Big[\frac{3xyz(x^2+y^2+z^2)^{\frac{1}{2}}}{2} + xy(x^2+y^2)\log\Big(\frac{z+(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}}\Big) + xz(x^2+z^2)\log\Big(\frac{y+(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+z^2)^{\frac{1}{2}}}\Big)$$
$$+ yz(y^2+z^2)\log\Big(\frac{x+(x^2+y^2+z^2)^{\frac{1}{2}}}{(y^2+z^2)^{\frac{1}{2}}}\Big) - \frac{x^4}{2}\tan^{-1}\Big(\frac{yz}{x(x^2+y^2+z^2)^{\frac{1}{2}}}\Big) - \frac{y^4}{2}\tan^{-1}\Big(\frac{xz}{y(x^2+y^2+z^2)^{\frac{1}{2}}}\Big)$$
$$-\frac{z^4}{2}\tan^{-1}\Big(\frac{xy}{z(x^2+y^2+z^2)^{\frac{1}{2}}}\Big)\Big]dxdydz,$$

$$= \frac{4}{3a^2b^2c^2}\int_0^a\int_0^b\Big[\frac{7xy(x^2+y^2)^{\frac{3}{2}}}{20} - \frac{7xy(x^2+y^2+c^2)^{\frac{3}{2}}}{20} + \frac{19c^2xy(x^2+y^2+c^2)^{\frac{1}{2}}}{20} + cxy(x^2+y^2)\log\Big(\frac{c+(x^2+y^2+c^2)^{\frac{1}{2}}}{(x^2+y^2)^{\frac{1}{2}}}\Big)$$
$$+\frac{3c^2x(2x^2+c^2)-3x(x^2+c^2)^2}{20}\log\Big(\frac{y+(x^2+y^2+c^2)^{\frac{1}{2}}}{(x^2+c^2)^{\frac{1}{2}}}\Big) + \frac{3c^2y(2y^2+c^2)-3y(y^2+c^2)^2}{20}\log\Big(\frac{x+(x^2+y^2+c^2)^{\frac{1}{2}}}{(y^2+c^2)^{\frac{1}{2}}}\Big)$$
$$+\frac{3x^5}{20}\log\Big(\frac{y+(x^2+y^2)^{\frac{1}{2}}}{x}\Big) + \frac{3y^5}{20}\log\Big(\frac{x+(x^2+y^2)^{\frac{1}{2}}}{y}\Big) - \frac{cx^4}{2}\tan^{-1}\Big(\frac{cy}{x(x^2+y^2+c^2)^{\frac{1}{2}}}\Big) - \frac{cy^4}{2}\tan^{-1}\Big(\frac{cx}{y(x^2+y^2+c^2)^{\frac{1}{2}}}\Big)$$
$$-\frac{c^5}{10}\tan^{-1}\Big[\frac{xy}{c(x^2+y^2+c^2)^{\frac{1}{2}}}\Big]\Big]dxdy, = \frac{4}{3a^2b^2c^2}\int_0^a\Big[\frac{x^6}{15} - \frac{x(x^2+b^2)^{\frac{5}{2}}}{15} + \frac{4b^2x(x^2+b^2)^{\frac{3}{2}}}{15} - \frac{b^4x(x^2+b^2)^{\frac{1}{2}}}{8}$$
$$-\frac{x(x^2+c^2)^{\frac{5}{2}}}{15} + \frac{4c^2x(x^2+c^2)^{\frac{3}{2}}}{15} - \frac{c^4x(x^2+c^2)^{\frac{1}{2}}}{8} + \frac{x(x^2+b^2+c^2)^{\frac{5}{2}}}{15} - \frac{4x(b^2+c^2)(x^2+b^2+c^2)^{\frac{3}{2}}}{15}$$
$$+\frac{x(b^2+c^2)^2(x^2+b^2+c^2)^{\frac{1}{2}}}{8} + \frac{2b^2c^2x(x^2+b^2+c^2)^{\frac{1}{2}}}{5} + \frac{3b^2cx(2x^2+b^2)-3cx(x^2+b^2)^2}{20}\log\Big[\frac{c+(x^2+b^2+c^2)^{\frac{1}{2}}}{(x^2+b^2)^{\frac{1}{2}}}\Big]$$
$$+\frac{3bc^2x(2x^2+c^2)-3bx(x^2+c^2)^2}{20}\log\Big[\frac{b+(x^2+b^2+c^2)^{\frac{1}{2}}}{(x^2+c^2)^{\frac{1}{2}}}\Big] - \frac{(b^2+c^2)(b^4-6b^2c^2+c^4)}{40}\log\Big[\frac{x+(x^2+b^2+c^2)^{\frac{1}{2}}}{(b^2+c^2)^{\frac{1}{2}}}\Big]$$
$$+\frac{3bx^5}{20}\log\Big[\frac{b+(x^2+b^2)^{\frac{1}{2}}}{x}\Big] + \frac{b^6}{40}\log\Big[\frac{x+(x^2+b^2)^{\frac{1}{2}}}{b}\Big] + \frac{3cx^5}{20}\log\Big[\frac{c+(x^2+c^2)^{\frac{1}{2}}}{x}\Big] + \frac{c^6}{40}\log\Big[\frac{x+(x^2+c^2)^{\frac{1}{2}}}{c}\Big]$$
$$-\frac{cx^4}{2}\tan^{-1}\Big[\frac{bc}{x(x^2+b^2+c^2)^{\frac{1}{2}}}\Big] - \frac{b^5c}{10}\tan^{-1}\Big[\frac{cx}{b(x^2+b^2+c^2)^{\frac{1}{2}}}\Big] - \frac{bc^5}{10}\tan^{-1}\Big[\frac{bx}{c(x^2+b^2+c^2)^{\frac{1}{2}}}\Big]\Big]dx,$$

$$= \frac{e}{15} + \frac{4[a^7+b^7+c^7+e^7-(a^2+b^2)^{\frac{7}{2}}-(a^2+c^2)^{\frac{7}{2}}-(b^2+c^2)^{\frac{7}{2}}]}{315a^2b^2c^2} + \frac{7}{90}\Big[\frac{(a^2+b^2)^{\frac{3}{2}}}{c^2} + \frac{(a^2+c^2)^{\frac{3}{2}}}{b^2} + \frac{(b^2+c^2)^{\frac{3}{2}}}{a^2}\Big]$$
$$-\frac{7e}{90}\Big[\frac{a^4+b^4}{a^2b^2} + \frac{a^4+c^4}{a^2c^2} + \frac{b^4+c^4}{b^2c^2}\Big] + \frac{1}{12}\Big[\frac{a^2+b^2}{c} - \frac{a^6+b^6}{5a^2b^2c}\Big]\log\Big[\frac{e+c}{e-c}\Big] + \frac{1}{12}\Big[\frac{a^2+c^2}{b} - \frac{a^6+c^6}{5a^2bc^2}\Big]\log\Big[\frac{e+b}{e-b}\Big]$$
$$+\frac{1}{12}\Big[\frac{b^2+c^2}{a} - \frac{b^6+c^6}{5ab^2c^2}\Big]\log\Big[\frac{e+a}{e-a}\Big] + \frac{a^4}{30bc^2}\log\Big[\frac{b+(a^2+b^2)^{\frac{1}{2}}}{a}\Big] + \frac{a^4}{30b^2c}\log\Big[\frac{c+(a^2+c^2)^{\frac{1}{2}}}{a}\Big]$$
$$+\frac{b^4}{30ac^2}\log\Big[\frac{a+(a^2+b^2)^{\frac{1}{2}}}{b}\Big] + \frac{b^4}{30a^2c}\log\Big[\frac{c+(b^2+c^2)^{\frac{1}{2}}}{b}\Big] + \frac{c^4}{30ab^2}\log\Big[\frac{a+(a^2+c^2)^{\frac{1}{2}}}{c}\Big]$$
$$+\frac{c^4}{30a^2b}\log\Big[\frac{b+(b^2+c^2)^{\frac{1}{2}}}{c}\Big] - \frac{2a^3}{15bc}\tan^{-1}\Big[\frac{bc}{ae}\Big] - \frac{2b^3}{15ac}\tan^{-1}\Big[\frac{ac}{be}\Big] - \frac{2c^3}{15ab}\tan^{-1}\Big[\frac{ab}{ce}\Big].$$

87.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Erie, Erie County, Pennsylvania.

A cask containing a gallons of wine is placed upon another containing b gallons of brandy. Water runs in at the top of the wine cask at the rate of m gallons per minute, the mixture escapes into the brandy cask at the same rate, the mixture in the brandy cask escapes at a like rate into a tub containing c gallons of water and the tub overflows. Find the quantity of each fluid in the tub at the end of t minutes, supposing them to mingle perfectly.

Solution by WILLIAM WOOLSEY JOHNSON, Professor of Mathematics, St. John's College, Annapolis, Anne Arundel Co., Md.

1. Let x denote the number of gallons of wine in the first cask at the end of t minutes, y the number of gallons in the second cask, and z the number of gallons in the tub.

The rate per minute at which wine is passing from the first into the second cask is $\frac{mx}{a}$; this fraction also represents the rate at which wine is entering the second cask, while $\frac{my}{b}$ is the rate at which wine is leaving it. Thus we have the differential equations:

$$\frac{dx}{dt}=-\frac{mx}{a}\text{.......(1)},\quad \frac{dy}{dt}=\frac{mx}{a}-\frac{my}{b}\text{.......(2)},\quad \frac{dz}{dt}=\frac{my}{b}-\frac{mz}{c}\text{.......(3)};$$

and to determine the constants of integration we have, when $t=0$, $x=a$, $y=0$, $z=0$. From (1), $\frac{dx}{x}=-\frac{mdt}{a}$; $\therefore\ x=e^{-\frac{mt}{a}}+$ constant, and, since $t=0$ gives $x=a$, $x=ae^{-\frac{mt}{a}}$......(4). Substituting this value of x, (2) becomes $\frac{dy}{dt}=me^{-\frac{mt}{a}}-\frac{my}{b}$........(5).

Since the solution would be of the form $y=Be^{-\frac{mt}{b}}$...(6) were the first term of (5) zero, we may assume the solution to be of the form (6), B denoting a function of t.

Differentiating (6), $\frac{dy}{dt}=e^{-\frac{mt}{b}}\left(\frac{dB}{dt}\right)-\frac{m}{b}Be^{-\frac{mt}{b}}$, which agrees with (5) if $\frac{dB}{dt}=me^{\frac{mt}{b}-\frac{mt}{a}}$...............(7); from which $B=\frac{ab}{a-b}e^{\frac{mt}{b}-\frac{mt}{a}}+k$; but, since $t=0$ gives $y=0$, and therefore, by (6), $B=0$, we have $k=-\frac{ab}{a-b}$, and substituting the value of B and k in (6) $y=\frac{ab}{a-b}\left(e^{-\frac{mt}{a}}-e^{-\frac{mt}{b}}\right)$............................(8).

Substituting this value of y, (3) becomes $\frac{dz}{dt}=\frac{ma}{a-b}\left(e^{-\frac{mt}{a}}-e^{-\frac{mt}{b}}\right)-\frac{mz}{c}$ ......(9), whence, as above, we assume $z=Be^{-\frac{mt}{c}}$.......(10); differentiating, $\frac{dz}{dt}=e^{-\frac{mt}{c}}\frac{dB}{dt}-\frac{m}{c}Be^{-\frac{mt}{c}}$, which agrees with (9) if

$$\frac{dB}{dt}=\frac{ma}{a-b}\left(e^{\frac{mt}{c}-\frac{mt}{a}}-e^{\frac{mt}{c}-\frac{mt}{b}}\right).$$

Integrating, $B=\frac{a^2c}{(a-b)(a-c)}e^{\frac{mt}{c}-\frac{mt}{a}}-\frac{abc}{(a-b)(b-c)}e^{\frac{mt}{c}-\frac{mt}{b}}+k$; but since $t=0$ gives $z=0$, and therefore $B=0$, we have $k=\frac{ac^2}{(a-c)(b-c)}$.

Substituting the value of B and k in (10),

$$z=ac\left(\frac{a}{(a-b)(a-c)}e^{-\frac{mt}{a}}-\frac{b}{(a-b)(b-c)}e^{-\frac{mt}{b}}+\frac{c}{(a-c)(b-c)}e^{-\frac{mt}{c}}\right)\text{.................. (11).}$$

This gives the wine in the tub; the brandy in the tub is found by the proper change of constants in (7) to be $\frac{bc}{b-c}\left(e^{-\frac{mt}{b}}-e^{-\frac{mt}{c}}\right)$; whence the water in the tub is

$$c-\frac{a^2c}{(a-b)(a-c)}e^{-\frac{mt}{a}}+\frac{b^2c}{(a-b)(b-c)}e^{-\frac{mt}{b}}-\frac{c(ac+bc-ab)}{(a-c)(b-c)}e^{-\frac{mt}{c}}.$$

2. The solution takes a different form when two or more of the constants a, b, c are equal. Thus if $b=a$, we have in place of (7), $\frac{dB}{dt}=m$; whence $B=mt+k$, in which $t=0$ gives $k=0$; therefore $y=mte^{-\frac{mt}{a}}$......(12). We have now, in place of equation (9,) $\frac{dz}{dt}=\frac{m^2t}{a}e^{-\frac{mt}{a}}-\frac{mz}{c}$; and assuming as before $z=Be^{-\frac{mt}{c}}$, $\frac{dB}{dt}=\frac{m^2t}{a}e^{\frac{mt}{c}-\frac{mt}{a}}$....(13), whence, integrating, $B=\frac{mct}{a-c}e^{\frac{mt}{c}-\frac{mt}{a}}-\frac{ac^2}{(a-c)^2}e^{\frac{mt}{c}-\frac{mt}{a}}+k$ in which $t=0$ gives $k=\frac{ac^2}{(a-c)^2}$; hence substituting we have $z=\frac{mct}{a-c}e^{-\frac{mt}{a}}-\frac{ac^2}{(a-c)^2}e^{-\frac{mt}{a}}+\frac{ac^2}{(a-c)^2}e^{-\frac{mt}{c}}$...(14).

3. But if $a=b=c$, (13) becomes $\frac{dB}{dt}=\frac{m^2t}{a}$, whence $B=\frac{m^2t^2}{2a}+k$, in which $t=0$ gives $k=0$; hence in this case $z=\frac{m^2t^2}{2a}e^{-\frac{mt}{a}}$ ....(15).

The expression for the brandy in the tub would now be identical with that for y in (12).

4. Again, if while a and b are unequal, $c=a$, after assuming $z=Be^{-\frac{mt}{a}}$, as in (10), we have $\frac{dB}{dt}=\frac{ma}{a-b}-\frac{ma}{a-b}e^{\frac{mt}{a}-\frac{mt}{b}}$; whence $B=\frac{mat}{a-b}+\frac{a^2b}{(a-b)^2}e^{\frac{mt}{a}-\frac{mt}{b}}+k$, and determining k as before,

$$z=\frac{mat}{a-b}e^{-\frac{mt}{a}}+\frac{a^2b}{(a-b)^2}e^{-\frac{mt}{b}}-\frac{a^2b}{(a-b)^2}e^{-\frac{mt}{a}} \qquad (16).$$

5. Finally, a and b being unequal, if $c=b$ the expression for z is found by interchanging a and b in (16), and the expression for the brandy is now obviously of the form given in (12), that is $mte^{-\frac{mt}{b}}$.

Expression (12) might have been found from (8) by making $b=a$ and evaluating by any of the usual processes for indeterminate forms; and in like manner (14) might have been deduced from (11), (15) from (14) and (16) from (11).

Prof. *Johnson* also gave an elaborate solution for the general case of n casks, which we reluctantly omit for want of space.

The solutions by Prof. *DeVolson Wood*, *E. B. Seitz*, Prof. *Trowbridge* and *Cyrus B. Haldeman* are all excellent, and we wish we had room to publish them.

LIST OF CONTRIBUTORS TO THE SENIOR DEPARTMENT.

The following persons have furnished solutions to the Problems indicated by the numbers:

E. B. Seitz, Greenville, O., 56, 57, 58, 59, 60, 62, 63, 67, 68, 73, 74, 77, 80, 81, 83, 86 and 87, and 40 of "Unsolved Problems"; Prof. F. P. Matz, King's Mountain, N. C., 56, 60, 62, 69, 70, 74, 78 and 84; Prof. DeVolson Wood, Hoboken, N. J., 59, 60, 62, 75, 78, 84 and 87; Walter S. Nichols, New York, N. Y., 57, 62 and 68, and 39 and 40 of "Unsolved Problems"; Henry Heaton, Sabula, Iowa, 73, 77, 79, 80 and 82; William Hoover, Bellefontaine, O., 62, 70, 79 and 83; Prof. H. T. J. Ludwig, Mt. Pleasant, N. C., 74, 83 and 86; Walter Siverly, Oil City, Pa., 72, 75 and 78; Prof. David Trowbridge, Waterburg, N. Y., 57, 69 and 87; Prof. W. W. Johnson, Annapolis, Md., 85 and 87; Rev. W. J. Wright, Cape May Point, N. J., 64 and 85; Prof. Benjamin Peirce, Cambridge, Mass., 66 and 71; Miss Christine Ladd, Baltimore, Md., 58 and 61; Prof. D. J. McAdam, Washington, Pa., 72 and 74; W. E. Heal, Wheeling, Ind., 56 and 57; George Eastwood, Saxonville, Mass., 56 and 81; I. H. Turrell, Cincinnati, O., 63; Prof. Daniel Kirkwood, Bloomington, Ind., 69; Dr. Joel E. Hendricks, DesMoines, Iowa, 67; T. P. Stowell, Rochester, N. Y., 70; David Wickersham, Wilmington, O., 84; Cyrus B. Haldeman, Ross, Butler Co., O., 87; Dr. David S. Hart, Stonington, Conn., 65; Dr. S. H. Wright, Penn Yan, N. Y., 59; O. H. Merrill, Mannsville, N. Y., 57.

The first prize is awarded to Prof. W. W. Johnson, Annapolis, Md., and the second prize to Prof. DeVolson Wood, Hoboken, N J.

PROBLEMS.

111.—Proposed by Gavin Shaw, Kemble, Ontario, Canada.

It is required to divide a given number a so that the continued product of all its parts shall be the greatest possible.

112.—Proposed by Robins Fleming, Readington, Hunterdon County, New Jersey.

Find 34 right-angled triangles having the same hypothenuse.

113.—Proposed by Frederick S. Samuels, Cerro Gordo, Inyo County, California.

What is the volume of a chip cut at an angle of 45 degrees to the center of a round log, radius r?

114.—Proposed by Dr. David S. Hart, M. A., Stonington, New London County, Connecticut.

To find two numbers such that their sum shall be a square, the sum of their squares a square, and if the cube of each be added to the square of the other the sums shall be equal.

115.—Proposed by Sylvester Robins, North Branch Depot, Somerset County, New Jersey.

There is a series of parallelopipeds whose dimensions differ from a perfect cube by one unit in only one of the edges. In every case the solid diagonal is an integer. Calling the one whose edges are 1, 2, 2 the *first* parallelopiped, it is required to find general expressions for the dimensions of the nth solid, and compute the length, breadth and thickness of the 30th one.

116.—Proposed by Marcus Baker, U. S. Coast Survey Office, Washington, D. C.; and Joseph B. Mott, Neosho, Newton Co., Mo.

Solve the equation $x^x=a$, and find the value of x when $a=800$.

117.—Proposed by Dr. Joel E. Hendricks, M. A., Editor of the *Analyst*, DesMoines, Polk County, Iowa.

Suppose a ball to be projected, in a horizontal direction, and due west, from a point vertically above the 40th degree of north latitude; and suppose the projectile velocity to be just sufficient to arrest its motion in space which resulted from the earth's motion on its axis, so that it will descend to the earth in a vertical plane corresponding with the meridian through the point of projection. And suppose, further, that the ball, under the circumstances, shall fall to the earth in 5 seconds. How far will the ball strike the earth *north* of the 40th parallel?

118.—Proposed by Winfield V. Jeffries, Instructor in Mathematics, Vermillion Institute, Hayesville, Ashland Co., O.

How many different combinations, each composed of n letters, can be formed from m letters, of which a are one letter, b are another and c are another?

119.—Proposed by W. E. Heal, Wheeling, Delaware County, Indiana.

Show how to trisect an angle, and how to construct two mean proportionals between two given straight lines, by means of the curves called the *conchoid* and the *cissoid*.

120.—Proposed by J. J. Sylvester, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

Prove that for all positive values of k less than unity the equation $(x+a)(x+b)=k(x+c)^2$ has two real roots.

121.—Proposed by E. J. Edmunds, B. S., New Orleans, Orleans County, Louisiana.

Prove that $\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{2}{n}\right)\cdot\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right)=(2\pi)^{\frac{n-1}{2}}\cdot n^{-\frac{1}{2}}$, n being a positive integer and Γ denoting the well known Eulerian integral.

122.—Proposed by Prof. David Trowbridge, M. A., Waterburg, Tompkins County, New York.

When the earth is in perihelion, suppose the sun's mass to be suddenly increased by a half of itself, or so that m' becomes $\frac{3}{2}m'$. Required the change in the elements of the terrestrial orbit.

123.—Proposed by Dr. David S. Hart, M. A., Stonington, New London County, Connecticut.

To find three whole numbers such that the sum of the squares of any two of them increased by the product of the same two shall be a rational square.

124.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

The first of two casks contained a gallons of wine, and the second b gallons of water; c gallons were drawn from the second cask, and then c gallons were drawn from the first cask and poured into the second, and the deficiency in the first supplied with c gallons of water; c gallons were then drawn from the first cask, and c gallons drawn from the second and poured into the first and the deficiency in the second cask supplied with c gallons of wine. Required the quantity of wine in each cask after n such operations as that described above.

125.—Proposed by Orlando D. Oathout, Read, Clayton County, Iowa.

What is the average thickness of a slab sawed at random from a round log?

126.—Proposed by Francis M. Priest, Bryan, Williams County, Ohio.

Divide unity into three such positive parts that if unity be added to each part the three sums shall be rational cubes.

127.—Proposed by Samuel Roberts, M. A., F. R. S., Member of the London Mathematical Society, London, England.

Two random points being taken within a circle (1) on opposite sides of a given diameter, (2) on the same side, (3) anywhere; find in each case the average radius of the concentric circle touched by the chord through them.

128.—Proposed by E. P. Norton, Allen, Hillsdale County, Michigan.

There is a circular fish pond surrounded by palisades, to the outside of which a horse is tethered. The length of the tether is equal to the circumference of the pond. Required the diameter of the pond, supposing the horse to have the liberty of grazing an acre of grass.

129.—Proposed by Reuben Davis, Bradford, Stark County, Illinois.

It is required to find three positive numbers, such that if each be diminished by the cube of their sum the three remainders will be rational cubes.

130.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

A circle is inscribed at random in a given semicircle. Find (1) the average area of the circle and (2) the chance that the circle does not exceed $\frac{1}{3}$ of the semicircle.

131.—Proposed by DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

A spherical homogeneous mass m, radius r, contracts by the mutual attraction of its particles to a radius nr; if the work thus expended be suddenly changed into heat, how many degrees F. will the temperature of the mass be increased, its specific heat being s and the heat uniformly disseminated?

132.—Proposed by Franklin P. Matz, M. E., M. S., Professor of Mathematics and Astronomy, King's Mountain High School, King's Mountain, Cleaveland County, North Carolina.

Three persons, A, B and C, are banished to a level circular island, diameter $2r$ feet. At the center of the island is a cylindric fort, diameter $2a$ feet. During a dark and foggy night B and C stray away from A, and from each other, and lie down to rest. At the first dawn of clear morning, while B and C are yet reposing, A looks around for them. Required the probability that A, without moving from his place of observation, can see both of his companions.

133.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

A duck swims across a river a rods wide, always aiming for a point in the bank b rods up stream from a point opposite the place she started from. The velocity of the current is v miles an hour, and the duck can swim n miles an hour in still water. Required the equation of the *curve* the duck describes in space, and the distance she swims in crossing the river.

134.—Proposed by WILLIAM HOOVER, Mathematical Editor of the *Wittenberger*, Bellefontaine, Logan County, Ohio.

Two equal rings begin to move freely from the extremity of the horizontal radius of a quadrant of a circle, one down the arc, the other down the chord; if the radius be 200 feet, how long after the motion begins will one ring be vertically above the other?

135.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Three equal circles touch each other externally; find the average area of all the circles that can be drawn in the space enclosed by them.

136.—Proposed by JOHN W. BERRY, Pittston, Luzerne County, Pennsylvania.

A smooth, straight, thin tube is balancing horizontally about its middle point, and a particle whose weight is $\frac{1}{n}$ that of the tube is shot into it horizontally with such a given velocity that it just arrives at the middle point of the tube. Find the angular velocity communicated to the tube.

137.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Two points are taken at random in the surface of a circle, but on opposite sides of a given diameter; find (1) the chance that the chord drawn through them does not exceed a line of given length, and (2) the average length of the chord.

138.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Two sides of a plane triangle are a and b; find its average area.

139.—Proposed by M. H. DOOLITTLE, U. S. Coast Survey Office, Washington, D. C.

Suppose infinitesimal aerolites equally distributed through all space, everywhere moving equally in all directions with a given uniform and constant absolute velocity. The aggregate mass intercepted in a given time by a given stationary sphere is supposed to be known. Determine the effect upon the eccentricity of a spherical planet of given mass and volume moving in an eccentric orbit, all of whose elements are given.

140.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

A circle is drawn intersecting a given circle, its center being at a given distance from that of the given circle. Find the average area common to both circles.

141.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

A row of particles in contact are charged some with positive and some with negative electricity. Under the effect of their mutual actions or the influence of some external body, these electricities are subject to *continuous* variation so conditioned that when the electricity of any intermediate particle becomes neutral the electricities of the particles on either side of it are of opposite signs.

Supposing the electricities of each of the particles to be known for one moment of time and for some subsequent moment, show that a certain logical inference can be drawn as to the joint changes undergone by any two extreme or any other two particles in the interval, and state what it is.

142.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

Two points are taken at random in the surface of an ellipse, one on each side of the major axis; find (1) the average distance between the points, and (2) the average length of the chord drawn through them.

143.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

A chord is drawn through two points taken at random in the surface of a circle; if a second chord be drawn through two other points taken at random in the surface, find the average area of the quadrilateral formed by joining the extremities of the chords.

144.—Proposed by E. B. SEITZ, Greenville, Darke County, Ohio.

A circle is circumscribed about a triangle formed by joining three points taken at random in the surface of a circumscriptible polygon of n sides; (1) find the chance that the circle lies wholly within the polygon; and (2), a second circle being described in the same manner, find the chance that both circles lie wholly within the polygon and one of the circles is wholly within the other.

145.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Middlesex County, Massachusetts.

Prove that if two bodies revolve about a center, acted upon by a force proportional to the distance from the center, and independent of the mass of the attracted body, each will appear to the other to move in a plane, whatever may be their mutual attraction.

146.—Proposed by Rev. W. J. WRIGHT, M. A., Ph. D., Member of the London Mathematical Society, Cape May Point, Cape May County, New Jersey.

If S_{ik} be the coefficient of a_{ik} in the determinant $D = \Sigma \pm a_{11}a_{22}...a_{nn}$, and Δ denote the determinant $\Sigma \pm S_{11}S_{22}...S_{nn}$, prove that $\Delta = D^{n-1}$.

147.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

A right cone, whose equation is $x^2+y^2=e^2z^2$...(1), and a paraboloid of revolution, whose equation is $y^2+z^2=px$...(2), have their vertices coincident, the axis of the cone being perpendicular to the axis of the paraboloid. Find the volume common to both by the formula $V=\int\int\int dxdydz$.

Solutions of these problems should be received by October 1, 1879.

EDITORIAL NOTES.

We regret to have to record the death of a valued and talented contributor, viz.: Dr. S. F. BACHELDER, of South Boston, Mass. The part of the following notice enclosed by quotation marks is condensed from a letter received from Mrs. Bachelder, dated March 2, 1878.

"Dr. Samuel F. Bachelder was born in the town of London, N H., on the 14th of October, 1829. He received an academical education at Gilmanton, N. H., studied medicine in his native town, and attended lectures at Cambridge, Mass. He settled in Peabody, Mass., where he had a good practice; but twelve years ago he moved to Boston, where he has since lived and practiced his profession until his death. He was a member of the Massachusetts Medical Society, and a Master Mason of St. Paul's Lodge, South Boston. He was much beloved by all who knew him for his kindly spirit and gentle bearing. Four years ago he began to have symptoms of spinal difficulty, which gradually increased and extended to the brain and caused his death January 1st, 1878."

Dr. Bachelder was an acceptable contributor to the VISITOR, and to the mathematical departments of the *National Educator* and the *Wittenberger*. He was very skillful and ingenious in the solution of algebraic and geometrical problems.

We tender our best thanks to Mr. E. B. SEITZ for valuable assistance, and for his successful efforts in extending the circulation of the VISITOR; and also to Mr. REUBEN KNECHT, of Easton, Pa., for the subscriptions secured by him.

The present No. has been delayed a month in consequence of the difficulty of procuring Greek type. We trust a like delay will not occur again.

No. 4 will be published about the first of January, 1880; it will contain about 32 pages and the price will be 50 cents. Persons desiring to secure copies should send their orders at an early date, as only a limited number will be printed.

Having reprinted No. 1, in fine style, we are now prepared to supply copies of that No. Price, 50 cents. The new edition contains an article on the Intrinsic Equation of a Curve, by the Editor, inserted to fill out the last leaf. We can also supply copies of No. 2 at 50 cents each.

If every one of our present subscribers would procure *one* new one, we could publish the VISITOR semi-annually. How many will secure the extra subscriber?

Send all orders to ARTEMAS MARTIN, Lock Box 11, Erie, Pa.

NOTICES OF BOOKS AND PERIODICALS.

The Elements of Analytical Mechanics. By DeVolson Wood, M. A., C. E., Professor of Mathematics in Stevens Institute of Technology. Second Edition. Revised, Corrected and Enlarged. New York: John Wiley & Sons. 8vo., cloth, pp. 249. Price $3.00.

The plan of the new edition is the same as the former one. "It is designed especially for students who are beginning the study of Analytic Mechanics, and is preparatory to the higher works upon the same subject, and to Analytical Physics and Astronomy. The Calculus is freely used."

The work is written in a remarkably clear and elegant style and beautifully printed, and illustrated with numerous diagrams. The general formulas are applied to the solution of a large number of interesting problems. In short, it is an excellent work upon the subject of which it treats, and we heartily commend it to all persons in want of an elementary treatise on this branch of mathematical science.

The Philosophy of Arithmetic as Developed from the three Fundamental Processes of Synthesis, Analysis, and Comparison. Containing also a History of Arithmetic. By Edward Brooks, Ph. D., Principal of Pennsylvania State Normal School, and Author of a Normal Series of Mathematics. 8vo., cloth, pp. 570.

This work is Dr. Brooks' crowning effort. It should be in the hands of every teacher in the land.

The chapters on the Origin and Development of Arithmetic, Early Writers on Arithmetic, Origin of Arithmetical Processes and the Origin of Arithmetical Symbols are deserving of special attention.

The whole book is written in Dr. Brooks' happiest style, and is as fascinating as a novel; it contains a vast fund of historical and other valuable information that the student can not find elsewhere without poring over many huge and musty volumes.

Plane Trigonometry and Functional Analysis. By Alfred H. Welsh, M. A., late Professor of Mathematics in the Akron Buchtel College, and present Instructor of Rhetoric and English Literature in the Columbus High School. 8vo., cloth, pp. 190. Columbus: G. J. Brand & Company.

A good elementary work on Logarithms, Plane Trignometry and Trignometrical Analysis. It is well printed on heavy tinted paper, but the composition is apparently the work of inexperienced hands.

Tracts Relating to the Modern Higher Mathematics. Tract No. 3. Invariants. By Rev. W. J. Wright, M. A., Ph. D., Member of the London Mathematical Society. In press.

The author has kindly sent us advance sheets of the first 54 pp. A perusal of them convinces us that the Tract will be a valuable addition to the literature of the subject.

The Normal Higher Arithmetic, Designed for Common Schools, High Schools, Normal Schools, Academies, etc. By Edward Brooks, A. M., Ph. D., Principal and Professor of Mathematics in Pennsylvania State Normal School. 12mo., pp. 514. Price $1.25. Philadelphia: Sower, Potts & Company.

A most excellent work; we unhesitatingly pronounce it the *best* Higher Arithmetic we have ever seen.

The Normal Union Arithmetic, Designed for Common Schools, Normal Schools, High Schools, Academies, etc. By Edward Brooks, A. M., Ph. D. 12mo., pp. 424. Philadelphia: Sower, Potts & Company.

A leading feature of the work is the union of mental and written arithmetic in one book—hence its name. A happy combination, and we predict for it a wide popularity. Just the thing for common schools.

Researches in Graphical Statics. By Henry T. Eddy, C. E., Ph. D., Professor of Mathematics and Civil Engineering in the University of Cincinnati. Reprinted from Van Nostrand's Engineering Magazine. 8vo., pp. 122.

Consists of several articles on Arches, Stresses and kindred subjects of special interest to the civil engineer.

On the Origin of Comets. By H. A. Newton, LL. D., Professor of Mathematics in Yale College. 8vo., pp. 15. From the American Journal of Science and Arts. Vol. XVI, Sept., 1878.

An interesting comparison of the theories of Kant and Laplace in regard to the origin of these wanderers in space. Kant believed them to be formed from the matter of the condensing solar nebula; Laplace considered that they were made from the matter that is scattered through the stellar spaces.

Prof. Newton concludes that the mass of observed facts are in favor of the hypothesis of Laplace.

Observations and Orbits of the Satellites of Mars. By Asaph Hall, Professor of Mathematics, U. S. Navy. 4to., pp. 46. Washington. 1878.

An interesting account of the important discovery which has immortalized the name of Professor Hall. Hamilton College, at its last commencement, conferred upon him the well-merited Honorary Degree of Ph. D.

The Educational Times and Journal of the College of Preceptors is published monthly by C. F. Hodgson & Son, London, England, and contains a valuable Mathematical Department of three or four double-column quarto pages, ably edited by W. J. C. Miller, B. A., Registrar of the General Medical Council, which numbers among its contributors many of the leading mathematicians of England, Continental Europe and this country.

Mathematical Questions, with their Solutions, Reprinted from the Educational Times. Same publishers. Issued in half-yearly volumes of 112 pp., 8vo., boards.
The *Reprint* contains, besides the mathematics published in the *Times*, many additional solutions and papers. Vol. XXVIII contains 8 papers and solutions of 71 problems; Vol. XXIX contains 10 papers and solutions of 91 problems.
Several of the solutions of problems in the Diophantine Analysis are by our able contributor, Dr. David S. Hart, M. A., and quite a number of those in "Average" and "Probability" are by our able contributor, Mr. E. B. Seitz.
The Editor of the VISITOR can furnish the *Times* at $2.00 a year, and the *Reprint* at $1.75 per volume.

The American Journal of Mathematics, Pure and Applied. Prof. J. J. Sylvester, LL. D., F. R. S., Editor in Chief; W. E. Story, Ph. D., (Leipsic), Associate Editor in Charge; with the co-operation of Benjamin Peirce, LL. D., F. R. S., of Harvard University; Simon Newcomb, LL. D., F. R. S., of the Naval Observatory; and H. A. Rowland, C. E. Published under the auspices of the Johns Hopkins University, Baltimore, Md. Issued in quarterly numbers of 96 quarto pages. Price $5.00 a year.
The first volume of this very able mathematical periodical is completed, and contains papers of the highest order of excellence.
The *Journal* numbers among its contributors many of the distinguished mathematicians of the world.

The Analyst. A Journal of Pure and Applied Mathematics. DesMoines, Iowa: Edited and published by J. E. Hendricks, M. A. Bi-monthly; each number contains 32 pp. $2.00 a year.
No. 1, Vol. VI, presents an interesting table of contents. Dr. Hendricks is deserving of great credit for the very able manner in which he has so successfully conducted the *Analyst*, and should have a remunerative subscription list.

The Wittenberger, a neat monthly magazine, devoted to the interest of Wittenberg College, published at Springfield, Ohio, has an excellent mathematical department edited by William Hoover. Price $1.10 a year.

Barnes' Educational Monthly. New York: A. S. Barnes & Co. $1.50 a year.
An excellent educational monthly. The mathematical department is under the management of, and very ably conducted by, Prof. F. P. Matz, M. E., M. S.

Educational Notes and Queries. Prof. W. D. Henkle, Salem, Ohio, Editor and Publisher. Issued monthly except in the vacation months of July and August. $1.00 a year.
Contains, besides a vast fund of other interesting matter, mathematical notes, problems and solutions.

The Maine Farmers' Almanac for 1879 contains, besides Puzzles, Riddles, &c., Mathematical Questions, and answers to those proposed last year. Hallowell, Me.: Masters & Livermore. Price 10 cents.

The New-England Journal of Education. Weekly. Boston: Published under the auspices of the AMERICAN INSTITUTE OF INSTRUCTION and the TEACHERS' ASSOCIATIONS of the New-England States. $2.50 a year.
The mathematical department is very efficiently conducted by Prof. E. T. Quimby, of Dartmouth College.

The Yates County Chronicle. Weekly. Penn Yan, N. Y.: Chronicle Publishing Co. $2.00 a year.
The mathematical department continues under the able management of Dr. S. H. Wright, M. A., Ph. D.

The Tunkhannock Republican. Weekly. Tunkhannock, Pa.: Cyrus D. Camp. $2.00 a year.
The *Republican* has an interesting arithmetical department edited by C. W. Bushnell.

The Pennsylvania School Journal. Monthly. J. P. Wickersham, Editor. Lancaster: J. P. Wickersham & Co. $1.60 a year.
The Pennsylvania School Journal is one of the best educational periodicals in the country. It has no mathematical department, but occasionally contains scientific articles and examination problems.

The American Journal of Education. Monthly. St. Louis: J. B. Merwin. $1.60 a year.
A very excellent "Journal of Education," and it should be liberally patronized by the teachers of Missouri and other States.

CORRIGENDA.

No. 2.

Page 15, line 3 from bottom, *for* "aces" *read* acres. Page 23, line 4, *for* "$(\frac{1}{1481})^n$" read $(\frac{4}{1481})^n$. Page 23, last line of solution of Problem 16, in part of the edition, *for* "$\frac{1}{4}$" *read* $\frac{1}{4^2}$. Page 32, line 7 from the bottom, *for* "$\cos^4(\theta+\phi)$" *read* $\cos^4\frac{1}{2}(\theta+\phi)$. Page 35, line 2 from the bottom, *for* "$\frac{2}{3\pi^2r^4}$" *read* $\frac{8}{3\pi^2r^4}$, and for "$\cos^4(\theta+\phi)$" *read* $\cos^4\frac{1}{2}(\theta+\phi)$. Page 37, line 4 from bottom, *for* "(13)" *read* (3). Page 38, line 14, *for* "(23)" *read* (13). Page 39, first line of Problem 14, *for* "r" *read* R; second line from bottom, *for* "$V=\pi aR^2$" *read* $V=\frac{\pi aR^2}{n}$, n being the number of cubic inches in a gallon. Page 40, change the signs of dQ. Page 41, line 12, in part of the edition, *for* "tan A log cosec A" *read* $\tan^2$A log cosec A. Page 42, lines 1 and 2, for "d^2" *read* d^3.

No. 3.

Page 55, solution of 49, last line but one, *for* "$\frac{3}{4}\sqrt{(119)}-\frac{142133}{133333}\sqrt{(119)}$" *read* $\frac{142133}{133333}\sqrt{(119)}-\frac{3}{4}\sqrt{(119)}$; solution of 50, line 2, *for* "$p_{a'}$" *read* p_a. Page 56, the sides of the fifth triangle should be 4059, 4060, 5741; and the sides of the sixth, 23660, 23661, 33461. Page 57, solution of 52, line 8, *for* "$\tan\frac{1}{2}\phi$" *read* $\tan\frac{1}{2}\omega$; in the value of BC *for* "$-b\sqrt{(bc)}$" *read* $-b\sqrt{(ac)}$. Page 59, line 3 of the value of N, in part of the edition *for* "-3728792" *read* -13728792; expunge b from the last term of line 8 of the value of N. Page 63, first solution of 60, last line, equation (6), *for* "$\text{in}\beta$" *read* $\sin\beta$. Page 65, second solution of 63, line 3, *for* "A′b" read A′B. Page 67, line 3, *for* "radius-rector" *read* radius-vector. Page 68, second solution of 70, line 6, in part of the edition, *for* "(R^2-x^2)" *read* $(R^2-x^2)^{\frac{1}{2}}$. Page 72, line 5 from the end of the solution of 78, *for* "t^2-t_2" *read* $t-t_2$. Page 73, solution of 80, line 23, *for* "ψ^2" *read* $12\psi^2$. Page 75, line 6, *for* "$-20R^2t^{11}$" *read* $-20Rr^2t^{11}$. In solution of 82 the value of p should be

$$\frac{\int_\theta^{\frac{1}{2}\pi} a\sec\theta\sin(\phi+\theta)\cos\phi\, d\phi}{\int_0^\theta 2a\tan\theta\cos^2\phi\, d\phi+\int_\theta^{\frac{1}{2}\pi} a\sec\theta\sin(\phi+\theta)\cos\phi\, d\phi},$$

and the value of q should be

$$\frac{\int_\theta^{\frac{1}{2}\pi} a\sec\theta\sin(\phi-\theta)\cos\phi\, d\phi}{\int_0^\theta 2a\tan\theta\cos^2\phi\, d\phi+\int_\theta^{\frac{1}{2}\pi} a\sec\theta\sin(\phi+\theta)\cos\phi\, d\phi}.$$

$= 4$

16

$R =$

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THE MATHEMATICAL VISITOR.

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Vol. I. *JANUARY, 1880.* *No. 4.*

JUNIOR DEPARTMENT.

Solutions of Problems Proposed in No. 3.

88.—Proposed by JOHN I. CLARK, Moran, Clinton County, Indiana.

When silver is 4 per cent. discount and gold is at 5 per cent. premium, taking greenbacks as a standard, what will silver be worth if we take gold as a standard?

I.—Solution by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; GAVIN SHAW, Kemble, Ont., Canada; WALTER S. NICHOLS, New York, N. Y.; ROBINS FLEMING, Readington, Hunterdon Co., N. J.; WILLIAM WILEY, Detroit, Mich.; Prof. H. S. BANKS, Instructor in English and Classical Literature, Newburg, Orange Co., N. Y.; and GEO. H. LELAND, Windsor, Windsor Co., Vt.

Silver being at 4 per cent. discount is worth 96; gold being at 5 per cent. premium is worth 105; hence silver is worth $\frac{96}{105}$ of gold, or $91\frac{3}{7}$.

II.—Solution by H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, Cabarrus Co., N. C.; THEO. L. DELAND, Office of the Secretary of the Treasury, Washington, D. C.; E. P. NORTON, Allen, Mich.; J. V. STEWART, Muncie, Delaware Co., Ind.; DAVID WICKERSHAM, County Surveyor, Wilmington, Clinton County, Ohio; and G. G. WASHBURN, North East, Erie County, Pennsylvania.

$\$1.00 - \$0.04 = \$0.96$, value of silver dollar in greenbacks; $\$1.00 + \$0.05 = \$1.05$, value of gold dollar in greenbacks; $\$\frac{100}{105}$ = value of greenback dollar in gold, and $\$0.96 \times \frac{100}{105} = \$0.91\frac{3}{7}$, value of silver dollar in gold.

III.—Solution by JOHN S. ROYER, Ansonia, Darke Co., O.; Prof. W. P. CASEY, San Francisco, California; W. T. MAGRUDER, Stevens Institute, Hoboken, N. J.; FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster County, Pa.; and V. W. HEATH, Rodman, Jefferson Co. N. Y.

Let 100 per cent. = value in greenbacks; then 96 per cent. = silver and 105 per cent. = gold. When gold is taken as the standard, silver will be $96 \div 1.05, = 91\frac{3}{7}$.

This problem was also correctly solved by *Thomas Bugot*, *A. R. Bullis*, *B. F. Burleson*, *J. R. Fagan*, Dr. *Hart*, Prof. *Matz*, *K. S. Putnam*, *J. A. Pollard*, *Sylvester Robins*, *T. P. Stowell*, *E. B. Seitz* and *Walter Siverly*.

89.—Proposed by G. G. WASHBURN, North East, Erie County, Pennsylvania.

Two men, A and B, each desire to sell a horse to C; A asks a certain price and B asks 50 per cent. more. C refused to pay either price; A then reduced his price $16\frac{2}{3}$ per cent. and B reduced his $33\frac{1}{3}$ per cent., at which prices C took both horses, paying \$220 for them. What was each man's asking price?

I.—Solution by Prof. W. P. CASEY, San Francisco, California; GAVIN SHAW, Kemble, Ontario, Canada; E. J. EDMUNDS, New Orleans, La.; and the PROPOSER.

A's asking price — $16\frac{2}{3}$ per cent. of it = $\frac{5}{6}$ of his asking price, and $\frac{3}{2}$ of A's asking price = B's asking price; but B's asking price — $33\frac{1}{3}$ per cent. of it = A's asking price, and therefore, per question, $\frac{5}{6}$ of A's asking price + once his asking price = \$220, or $\frac{11}{6}$ of his asking price = \$220; whence A's asking price = \$120 and B's = \$180.

II.—Solution by D. W. K. MARTIN, Webster, Darke County, Ohio; D. B. O'CONNER, Union City, Randolph County, Indiana; A. R. BULLIS, Ithaca, N. Y.; and J. S. ROYER, Ansonia, O.

100 per cent. = A's asking price, and 150 per cent. = B's. 100 per cent. — 16⅔ per cent. = 83⅓ per cent. = A's selling price; 150 per cent. — 33⅓ per cent. of 150 per cent. = 100 per cent., B's selling price. Then 83⅓ per cent. + 100 per cent. = \$220. 100 per cent. = **\$120**, A's asking price, and 150 per cent. = **\$180**, B's asking price.

Solved in a similar manner by *John I. Clark*, *Robins Fleming* and Prof. *E. B. Seitz*.

III.—Solution by GEORGE A. JOPLIN, Centre College, Danville, Ky.; Dr. DAVID S. HART, M. A., Stonington, New London Co., Conn.; WALTER SIVERLY, Oil City, Pa.; K. S. PUTNAM, Rome, N. Y.; THOMAS BAGOT, Principal Shelby School, Canaan, Indiana; T. P. STOWELL, Rochester, N. Y.; Prof. F. P. MATZ, M. E., Reading, Pa.; O. D. OATHOUT, Luana, Iowa; J. R. FAGAN, Erie, Pa.; and B. F. BURLESON, Oneida Castle, N. Y.

Let x = price A asked; then $\frac{3}{2}x$ = what B asked, $\frac{5}{6}x$ = A's reduced price and x = B's.
∴ $\frac{5}{6}x + x = \$220$, or $11x = \$1320$; whence $x = \$120$, A's asking price, and $\frac{3}{2}x = \$180$, B's asking price.

Solutions similar to the above were given by Messrs. *Albert*, *Banks*, *Drummond*, *Leland*, *Norton*, *Nichols*, *Pollard*, *Stewart*, *Shidy*, *Wickersham* and *Wiley*.

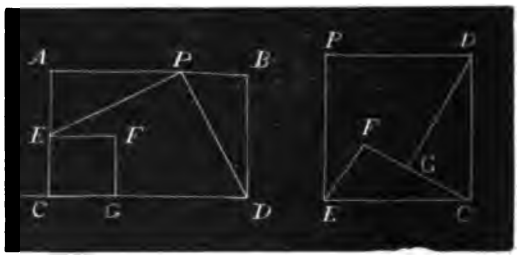
IV.—Solution by THEO. L. DELAND, Office of the Secretary of the Treasury, Washington, D. C; WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor *Wittenberger*, Wapakoneta, Ohio; and SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

Let $6x$ = A's asking price and $9x$ = B's; then $5x$ = A's selling price and $6x$ = B's.
∴ $6x + 5x = 11x = \$220$, and $x = \$20$. $6x = \$120$, A's asking price, and $9x = \$180$, B's asking price.

90.—Proposed by K. S. PUTNAM, Rome, Oneida County, New York.

Having cut a square, area a^2, from one corner of a board $2a$ wide and $3a$ long, cut the remainder into three pieces so that they will make a sqare.

Solution by B. F. BURLESON, Oneida Castle, Oneida County, N. Y.; JULIAN A. POLLARD, Windsor, Windsor County, Vermont; SYLVESTER ROBINS, North Branch Depot, Somerset Co., N. J.; G. H. LELAND, Windsor, Windsor Co., Vt.; and ROBINS FLEMING, Readington, Hunterdon County, New Jersey.



Let ABCD represent the board, AB being $3a$ units in length and BD $2a$ units; also let EFCG represent the square, whose side is a units in length, which is cut from the board. The area of the remaining piece ABEFGD $= 5a^2$, therefore the side of its equivalent square must be $a\sqrt{5}$ units in length.

Trisect the side AB at the point P, distant $2a$ units from the angle A; join DP and EP, and the remainder of the board will be divided into the three required pieces.

Demonstration.—The angle DPE is a right angle and each of the lines DP and EP is $a\sqrt{5}$ units in length; therefore they must constitute two sides of the required square, and the two extraneous triangles EAP and DBP when properly arranged must complete it. The arrangement of the pieces is shown in the right-hand figure in the diagram.

Solved in like manner by Messrs. *Albert*, *Bullis*, *Clark*, *Casey*, *Heath*, *Putnam*, *Seitz*, *Shidy* and *Siverly*.

91.—Proposed by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Pa.

Show that in any isosceles triangle, the square of a line drawn from the vertex to any point in the base, plus the product of the segments of the base, is constant.

I.—Solution by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

In the isosceles triangle ABC, AC = BC. Draw AD perpendicular to and bisecting the base BC. Also draw AE dividing BC into two unequal parts BE, EC; then (*Euclid* II, 5) $BD \times DC = BE \times EC + (ED)^2$; but $(ED)^2 = (AE)^2 - (AD)^2$, whence $BD \times DC = BE \times EC + (AE)^2 - (AD)^2$, and by transposition we get $(AD)^2 + BD \times DC = (AE)^2 + BE \times EC$; therefore the truth of the theorem is manifest.

Solved in a similar manner by Prof. *Banks*, *K. S. Putnam*, *Walter Siverly* and *Gavin Shaw*.

II.—Solution by Prof. E. J. EDMUNDS, B. S., Principal of Academic School No. 3, New Orleans, La.; J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; CHARLES H. TUTTON, Wilkes Barre, Luzerne County, Pa.; and T. P. STOWELL, Rochester, N. Y.

We are to prove that $(AE)^2 + BE \times EC$ = constant. Let AD be the altitude; we have $BE = BD - ED$; $EC = BD + ED$, for $BD = DC$; multiplying, $BE \times EC = (BD)^2 - (ED)^2$. Adding $(AE)^2$ to both members, $(AE)^2 + BE \times EC = (AE)^2 + (BD)^2 - (ED)^2 = (AD)^2 + (BD)^2 = (AB)^2 = (AC)^2$.

Solved in a similar manner by *Robins Fleming*, *E. P. Norton*, *J. A. Pollard*, *Sylvester Robins*, *E. B. Seitz* and *L. P. Shidy*.
Excellent solutions received from Messrs. *Albert*, *Baker*, *Bullis*, *Casey*, *Clark*, *Drummond*, *DeLand*, *Martin*, *Nichols*, *Oathout*, *Wickersham* and *Wiley*.

92.—Proposed by JAMES Q. BRIGHAM, Walton, Harvey County, Kansas.

What is the rate per cent. of interest when a sum of money amounts to ten times itself in 21 years, compounded annually? And what would be the rate for the same time if compounded semi-annually?

I.—Solution by WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Auglaize County, Ohio.

1.—Put $10 = n$, $21 = t$, the principal $= p$, and let $r =$ the rate per cent. required; then $p(1+r)^t = np$, or $(1+r)^t = n$. Taking logarithms, $t\log(1+r) = \log n$; whence $r = \log^{-1}\left(\frac{\log n}{t}\right) - 1, = 0.1158+$.

2.—Let $r_1 =$ the rate in this case; then $p(1+\frac{1}{2}r_1)^{2t} = np$, or $(1+\frac{1}{2}r_1)^{2t} = n$; whence

$$r_1 = 2\log^{-1}\left(\frac{\log n}{2t}\right) - 2, = 0.1127+.$$

II.—Solution by DAVID WICKERSHAM, County Surveyor, Wilmington, Clinton County, O.; Hon. J. H. DRUMMOND, Portland, Maine; K. S. PUTNAM, Rome, N. Y.; SYLVESTER ROBINS, North Branch Depot, N. J.; A. R. BULLIS, Ithaca, N. Y.; V. WEBSTER HEATH, Rodman, N. Y.; WILLIAM WILEY, Detroit, Mich.; G. H. LELAND, Windsor, Vt.; GAVIN SHAW, Kemble, Ontario, Canada; and JOHN I. CLARK, Moran, Clinton County, Indiana.

Let x be the rate per cent., and suppose the principal to be one dollar; then $(1+x)^{21} = 10$, and $\log(1+x) = \frac{1}{21} = 0.047619048$. The number answering to the logarithm 0.047619048 is 1.11588; therefore $1+x = 1.11588$, and the rate per cent. is 0.11588.

The rate per cent. when compounded semi-annually is found thus: Let y be the rate for six months and $(1+y)^{42} = 10$ and $\log(1+y) = \frac{1}{42} = 0.023809524$; the number answering to this logarithm is 1.05635, and the rate per cent. for six months is 0.05635, and the rate per cent. for a year is double this or 0.1127.

Nearly thus were the solutions by Messrs. *Banks*, *Casey*, *Fleming*, *Hart*, *Nichols*, *Oathout*, *Pollard*, *Royer*, *Shidy*, *Siverly* and *Seitz*.

93.—Proposed by Prof. E. J. EDMUNDS, B. S., New Orleans, Orleans County, Louisiana.

A point P being given on the base of a triangle, draw a line across the triangle parallel to the base which will subtend a right angle at P.

Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin Co., Pa.; and T. P. STOWELL, Rochester, Monroe Co., N. Y.

Construction.—Let ABC represent the given triangle, P the given point in the base. Draw the medial line CD, connect P with C, describe from D with a radius = BD an arc which cuts CP produced at E, join D with E, draw PF parallel to DE and through F draw GK parallel to AB; then GK is the required line.

Demonstration.—DE being parallel to FP, we have DE : FP = CD : CF; but GK being parallel to AB, BD : FK = CD : CF; hence DE : FP = BD : FK, but by construction DE = BD, therefore FP = FK = GF, consequently the semicircle described upon GK passes through P and the angle GPK is a right angle.

Also constructed in an ingenious manner by Prof. *Frank Albert*, Pof. *E. B. Seitz* and *L. P. Shidy*. Solved algebraically by Messrs. *Casey*, *Baker*, *Bullis*, *Hoover*, *Pollard*, *Siverly*, *Wiley* and *Wickersham*.

94.—Proposed by HENRY NICHOLS, Hampton, Rock Island Co., Ill.; and Mrs. ANNA T. SNYDER, Chicago, Cook Co., Ill.

It is required to divide a tapering board into two equal parts by sawing it across, parallel to the ends. Find the width at each end so that the lengths of the pieces will be expressed by rational numbers.

I.—Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.

Denoting AB by x, CD by y, EF by z, we have $\triangle ABG : \triangle DCG = x^2 : y^2$, or $ABCD : \triangle DCG = x^2 - y^2 : y^2 \ldots (1)$, and similarly $\triangle DCG : EFCD = y^2 : z^2 - y^2 \ldots (2)$. Multiplying (1) by (2), and taking into consideration that ABCD = 2EFCD, we obtain

$$x^2 + y^2 = 2z^2 \ldots\ldots (3).$$

In order to find rational numbers for x, y, z, we put $x = z + 2m$, $y = z - 2(m+1)$, and then find $z = 2m^2 + 2m + 1$, $x = 2m^2 + 4m + 1$, $y = 2m^2 - 1$, in which any integers may be substituted for m. Let $m = 1$, then $z = 5$, $x = 7$, $y = 1$. If $m = 2$, $z = 13$, $x = 17$, $y = 7$; &c., &c.

II.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania; and SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

Let a be the width of wide end, b the width of narrow end, $l =$ length of the board and $x =$ width where cut.

Then $\frac{l(a-x)}{a-b} =$ length of shorter piece, and $\frac{l(x-b)}{a-b} =$ length of longer piece, which will be rational when x is rational.

As the pieces are equal, we have $\dfrac{l(a-x)(a+x)}{2(a-b)}=\dfrac{l(x-b)(x+b)}{2(a-b)}$; whence $x=\frac{1}{2}\sqrt{(2a^2+2b^2)}$.

Put $a=m+n$ and $b=m-n$, and we have $x^2=m^2+n^2$. Now put $m=2pq$ and $n=p^2-q^2$, and we have $x=p^2+q^2$, $a=2pq+p^2-q^2$, $b=2pq-p^2+q^2$.

Taking $p=2$ and $q=1$, $x=5$, $a=7$, $b=1$; $p=3$ and $q=2$, $x=13$, $a=17$, $b=7$; etc., etc.

Another solution, by Dr. *David S. Hart, M. A.*, will be given in No. 5.
Good solutions received from Messrs. *Albert, Bullis, Casey, Clark, Drummond, Fleming, Hoover, Pollard, Seitz, Shidy, Tutton* and *Wickersham*.

95.—Proposed by T. A. Kinney, St. Albans, Franklin County, Vermont.

Given two circles and a point without them to draw through the point a line cutting the two circles so that the portion of the line intercepted between the two circles shall be equal to a given line.—[From *Chauvenet's Geometry*.

Solution by Marcus Baker, U. S. Coast Survey Office, Washington, D. C.

This problem which appears as Exercise 85 of *Chauvenet's Geometry* crept into that work through some mistake. It is there erroneously considered as a problem that can be *constructed* and is stated to admit of *two solutions*. It is a constant source of trouble to persons meeting it there without knowing its true character. It was proposed, in different words, in the *Analyst*, vol. ii, p. 156, Sept. 1875, by Prof. James G. Clark of Liberty, Mo., and by a curious coincidence appears as Problem 85. In the next number of the *Analyst*, p. 193, the true character of the problem was pointed out by Prof. Scheffer and myself. Prof. Scheffer stated that the problem admitted of *four* solutions while the equation obtained by myself was of the 8th degree. The following investigation however indicates neither *eight* nor *four* solutions, but *six*.

It is of interest to note that if we replace the two circles by two right lines perpendicular to each other we still have a difficult problem and one transcending the elements of Geometry. If however the given point lies in the bisector of the right angle the problem can be solved by the elements of Geometry. It is known as Pappus' Problem. Solutions of it may be found in several books. [See Catalan's Theoremes et Problemes de Geometrie elementaire, 8°, *Paris*, 1872, p. 146.

Let O and O′ be the centers of the two given circles whose radii are OM $=R$ and O′N $=r$, MN = the given line $=2l$, P the given point, x the distance from P to the middle of MN, $d=$ OP, $d'=$ O′P, $\angle$OPO′ $=\alpha$, $\angle$OPM $=\varphi$, $\angle$O′PN $=\varphi'$.

Now $\alpha+\varphi+\varphi'=180^\circ$, therefore $\cos\varphi=-\cos(\alpha+\varphi')$ or

$$\cos\varphi+\cos\alpha\cos\varphi'=\sin\alpha\sin\varphi';$$

whence squaring and transposing

$$\cos^2\varphi+\cos^2\varphi'+2\cos\alpha\cos\varphi\cos\varphi'-\sin^2\alpha=0 \ldots\ldots\ldots\ldots (1).$$

From the figure $\cos\varphi=\dfrac{(l-x)^2+d^2-R^2}{2d(l-x)}$ and $\cos\varphi'=\dfrac{(l+x)^2+d'^2-x^2}{2d'(l+x)}$.

For brevity put $d^2-R^2=m^2$ and $d'^2-r^2=n^2$, and substituting in (1) we have

$$\frac{m^4+2m^2(l-x)^2+(l-x)^4}{4d^2(l-x)^2}+\frac{n^4+2n^2(l+x)^2+(l+x)^4}{4d'^2(l+x)^2}+\frac{2[m^2+(l-x)^2][n^2+(l+x)^2]\cos\alpha}{4dd'(l^2-x^2)}-\sin^2\alpha=0,$$

from which equation x may be determined.

A somewhat laborious reduction reduces this equation to

$$Ax^6+Bx^5+Cx^4+Dx^3+Ex^2+Fx+G=0,$$

in which $A=d'^2-2dd'\cos\alpha+d^2$, $B=-2l(d'^2-d^2)$, $C=(2m^2-l^2)d'^2+(2n^2-l^2)d^2+2(3l^2-m^2-n^2)dd'\cos\alpha-4d^2d'^2\sin^2\alpha$, $D=4l[l^2(d'^2-d^2)+dd'(n^2-m^2)\cos\alpha]$, $E=(m^4-4m^2l^2-l^4)d'^2+(n^4-4n^2l^2-l^4)d^2-2(m^2n^2+3l^4)dd'\cos\alpha+8l^2d^2d'^2\sin^2\alpha$, $F=2l[(m^4-l^4)d'^2-(n^4-l^4)d^2+2l^2(m^2-n^2)dd'\cos\alpha]$,
$G=l^2\{d'^2(l^2+m^2)^2+d^2(l^2+n^2)^2+2[m^2n^2+(m^2+n^2)l^2+l^4]dd'\cos\alpha-4l^2d^2d'^2\sin^2\alpha\}$.

This problem was also solved by Prof. *Casey* and *Walter Siverly*. Prof. *Casey* employs trigonometric functions, and has had his solution published in the *Analyst*, vol. vi, p. 189. He remarks—"I do think it very likely the problem Prof. Chauvenet meant is this: 'Through one of the points of intersection of two given circles, to draw', &c., and not that given in his Geometry."

96.—Proposed by L. C. Walker, New Madison, Darke County, Ohio.

A cannon ball, radius r, rolls into the corner of a room whose walls are at right angles, and perpendicular to the floor. What is the radius of another ball just touched by the cannon ball?

Solution by John S. Royer, Ansonia, Darke County, Ohio; Prof. H. T. J. Ludwig, Mount Pleasant, N. C.; Prof. W. P. Casey, San Francisco, Cal.; Prof. J. F. W. Scheffer, Mercersburg, Pa.; Prof. E. B. Seitz, Kirksville, Mo.; Prof. Frank Albert, Millersville, Pa.; Hon. J. H. Drummond, LL. D., Portland, Me.; L. P. Shidy, U. S. Coast Survey Office, Washington, D. C.; K. S. Putnam, Rome, N. Y.; Robins Fleming, Readington, N. J.; C. H. Tutton, Wilkes Barre, Pa.; W. E. Heal, Wheeling, Ind.; J. A. Pollard, Windsor, Vt.; and G. H. Leland, Windsor, Vt.

Let $x=$ radius of smaller ball. The distance from the corner of the room to the center of the larger ball is the diagonal of a cube whose edge is r, and is equal to $r\sqrt{3}$. Similarly, the distance from the corner to the center of the smaller ball is $x\sqrt{3}$; then $x\sqrt{3}+x+r$ is the distance from the corner to the center of the larger ball. $\therefore x\sqrt{3}+x+r=r\sqrt{3}$; whence $x=r(2-\sqrt{3})$.

Solved also by Messrs. *Baker*, *Banks*, *Bullis*, *Clark*, *Hart*, *Heath*, *Hoover*, *Nichols*, *Norton*, *Matz*, *Siverly*. *Wickersham* and *Wiley*.

97.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Solve, by quadratics, the equation $x^4-2ax^3-2abx+b^2=0$.

I.—Solution by Prof. W. P. CASEY, San Francisco, California; E. A. BOWSER, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, N. J.; E. B. SEITZ, Professor of Mathematics, North Western State Normal School, Kirksville Mo.; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; Prof. H. S. BANKS, Newburg, N. Y.; SYLVESTER ROBINS, North Branch Depot, N. J.; and W. L. HARVEY, Maxfield, Maine.

Adding $(a^2+2b)x^2$ to each side, $x^4-2ax^3+(a^2+2b)x^2-2abx+b^2=(a^2+2b)x^2$.

Extracting the square root, we get $x^2-ax+b=\pm x\sqrt{(a^2+2b)}$, and $x^2-[a\pm\sqrt{(a^2+2b)}]x=-b$;

$$\therefore x=\tfrac{1}{2}\{a\pm\sqrt{(a^2+2b)}\pm\sqrt{[2a^2-2b\pm2a\sqrt{(a^2+2b)}]}\}.$$

Similarly solved by Messrs. *Hart*, *Leland*, *Pollard* and *Putnam*.

II.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let $x^4-2ax^3-2abx+b^2=(x^2+mx+b)(x^2+nx+b)$, $=x^4+(m+n)x^3+(2b+mn)x^2+b(m+n)x+b^2$.

Equating coefficients of like powers of x, we have $m+n=-2a$ and $mn=-2b$, which give

$$m=-a+\sqrt{(a^2+2b)},\quad n=-a-\sqrt{(a^2+2b)},$$

and we have the two quadratics $x^2-[a-\sqrt{(a^2+2b)}]x+b=0$, $x^2-[a+\sqrt{(a^2+2b)}]x+b=0$;

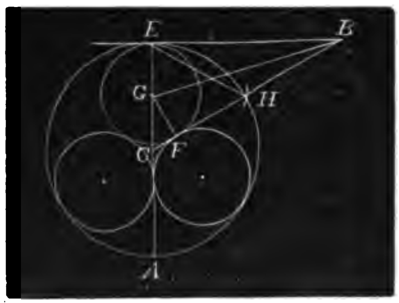
$$\therefore x=\tfrac{1}{2}\{a\mp\sqrt{(a^2+2b)}\pm\sqrt{[2a^2-2b\mp2a\sqrt{(a^2+2b)}]}\}.$$

Other solutions will be published in No. 5.
Good solutions given by Messrs. *Albert*, *Baker*, *Brown*, *Bullis*, *Heal*, *Hoover*, *Ludwig*, *Matz*, *Schæffer*, *Tullon* and *Wood*.

98.—Proposed by Mrs. ANNA T. SNYDER, Chicago, Cook County, Illinois.

On a circular griddle 12 inches in diameter, 3 equal circular cakes are baked of such size that each cake touches the edge of the griddle and the edges of the other two cakes. What is the diameter of each cake?

I.—Solution by the PROPOSER.



Let C be the center of the griddle, and draw any diameter AE. With center E and radius equal CE describe an arc intersecting the circumference of the griddle in H. Draw a tangent to the griddle at E and produce the radius CH to intersect the tangent at B. Bisect the angle EBC and draw the bisector; it will meet the radius CE in G, the center of one of the cakes. Draw GF perpendicular to CB; then GF = GE, the radius of one of the cakes.

By similar triangles, CF : FG :: CE : EB. But, if R = radius of griddle and r = radius of each cake, FB = EB = $R\sqrt{3}$ and CF = $2R-R\sqrt{3}$; $\therefore R(2-\sqrt{3}):r::R:R\sqrt{3}$; whence we get $r=R(2\sqrt{3}-3)$, and $2r=12(2\sqrt{3}-3)=5.5692+$ inches.

Elegant constructions also furnished by Prof. *Schæffer* and *L. P. Shidy*.

II.—Solution by E. P. NORTON, Allen, Hillsdale Co., Mich.; J. S. ROYER; Prof. FRANK ALBERT; Prof. E. J. EDMUNDS; Prof. J. F. W. SCHEFFER; WILLIAM HOOVER; J. V. STEWART; GAVIN SHAW; A. R. BULLIS; WILLIAM WILEY; J. A. POLLARD; ROBINS FLEMING; and SYLVESTER ROBINS.

Let $R=6$ inches, the radius of the griddle, and r = radius of each cake. Then $\tfrac{2}{3}r\sqrt{3}$ = distance from the center of the griddle to the center of each cake, and we have

$$r+\tfrac{2}{3}r\sqrt{3}=R.\quad\therefore r=R(2\sqrt{3}-3),\ \text{and}\ 2r=2R(2\sqrt{3}-3),\ =12(2\sqrt{3}-3),\ =5.5692+\ \text{inches}.$$

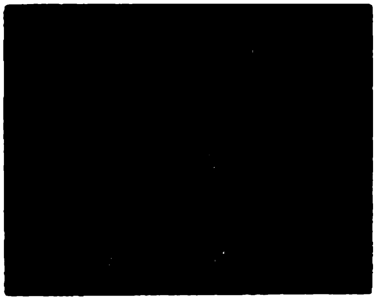
Excellent solutions received from Messrs. *Baker*, *Banks*, *Bowser*, *Casey*, *Clark*, *Drummond*, *Hart*, *Harvey*, *Heath*. *Leland*, *Martin*, *Nichols*, *Oathout*, *Seitz* and *Siverly*.

99.—Proposed by W. WOOLSEY JOHNSON, Professor of Mathematics, St. Johns College, Annapolis, Anne Arundel Co., Md.

The extremities of a line of fixed length which slides along a fixed line are joined to two fixed points. Find the locus of the intersection of the joining lines.

I.—Solution by ENOCH BEERY SEITZ, Professor of Mathematics, North Western State Normal School, Kirksville, Adair Co., Missouri.

Let HK be the line of fixed length, OS the fixed line along which HK slides, A and B the two fixed points, and P the intersection of AH and BK.



Produce AB till it meets OS in C; lay off CE = HK, CD = AB, and produce CA, making AN = BC. Draw BR, AT, NO, PM parallel to DE.

Let $OT = CR = a$, $AT = b$, $HK = CE = TR = c$, $OM = x$ and $PM = y$. Then we have $OR = CT = a + c$, $BR = \frac{ab}{a+c}$.

By similar triangles, BR − PM : OM − OR :: PM : MK, whence $MK = \frac{y(x-a-c)(a+c)}{ab-(a+c)y}$; and AT − PM : OM − OT :: PM : MH, whence $MH = \frac{y(x-a)}{b-y}$. $\therefore MK - MH = \frac{y(x-a-c)(a+c)}{ab-(a+c)y} - \frac{y(x-a)}{b-y} = c$, or by reduction, $xy = ab$, the equation to the required locus, which is, therefore, an hyperbola whose asymptotes are OS and ON.

A solution by Prof. *Johnson* will appear in No. 5.
The problem was also solved by Messrs. *Albert*, *Baker*, *Brown*, *Bullis*, *Casey*, *Edmunds*, *Hoover*, *Ludwig*, *Siverly* and *Tvllon*.

100.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Monroe County, Indiana.

Required to find a number such that when it is added to 15, 27 and 45 there arise three numbers which are in geometrical progression.

Solution by THOMAS BAGOT, Principal Shelby School, Canaan, Jefferson County, Indiana; THEO. L. DELAND, Office of the Secretary of the Treasury, Washington, D. C.; Prof. H. S. BANKS, Newburg, N. Y.; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; Prof. W. P. CASEY, San Francisco, California; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; D. B. O'CONNER, Union City, Ind.; W. L. HARVEY, Maxfield, Me.; WILLIAM WILEY, Detroit, Mich.; and J. S. ROYER, Ansonia, Darke County, Ohio.

Let x = the number; then $15 + x : 27 + x :: 27 + x : 45 + x$. This gives the equation

$$675 + 60x + x^2 = 729 + 54x + x^2; \text{ whence, } 6x = 54 \text{ and } x = 9.$$

Solutions similar in substance to the above were given by Messrs. *Albert*, *Bowser*, *Bullis*, *Clark*, *Drummond*, *Edmunds*, *Fleming*, *Hart*, *Heal*, *Hoover*, *Leland*, *Matz*, *Magruder*, *Norton*, *Oathout*, *Pollard*, *Putnam*, *Robins*, *Seitz*, *Shaw*, *Siverly* and *Tutton*.

101.—Proposed by O. H. MERRILL, Mannsville, Jefferson County, New York.

Two equal circles intersect, the center of each being on the circumference of the other. From A, one of the points of intersection, a diameter AB of one of the circles is drawn. Find the radius of the circle touching AB and the circumferences of both the given circles.

Solution by E. B. SEITZ, Professor of Mathematics, North Western State Normal School, Kirksville, Missouri.

Let M and N be the centers of the given circles. There may be three circles drawn to touch AB and the given circles, as shown in the diagram.

To find the radius of the circle O, let $AM = r$, $OP = x$. Then $OM = r - x$, $MP = \sqrt{(r^2 - 2rx)}$, $AP = r - \sqrt{(r^2 - 2rx)}$, and we have the equation

$$x = [r - \sqrt{(r^2 - 2rx)}]\tan 30^\circ, \text{ whence } x = \tfrac{2}{3}r(\sqrt{3} - 1).$$

To find the radii of the circles O′ and O″, let O′R or O″S = y, ∠O′MR or ∠O″MS = θ, ∠NMO′ or ∠NMO″ = φ. Then $\theta + \varphi = 180^\circ \mp 60^\circ$; hence $\sin(\theta + \varphi) = \pm\tfrac{1}{2}\sqrt{3}$, whence by developing and reducing we get

$$\sin^2\theta + \cos^2\varphi \mp \sqrt{3}\sin\theta\cos\varphi = \tfrac{1}{4}.$$

But $\sin\theta = \frac{y}{r-y}$, and $\cos\varphi = \frac{r-4y}{2(r-y)}$. Substituting the values of $\sin\theta$ and $\cos\varphi$, and solving, we find the radii to be $O'R = \frac{1}{169}r(66 - 10\sqrt{3})$, and $O''S = \frac{1}{169}r(66 + 10\sqrt{3})$.

No other complete solution received. Partial solutions given by Messrs. *Albert*, *Bullis*, *Casey*, *Hoover*, *Schaffer*, *Siverly*, *Tutton* and *Wiley*.

102.—Proposed by J. D. WILLIAMS, Superintendent of Public Schools, Sturgis, St. Joseph County, Michigan.

Solve the equations $x + y + xy = 75$, $x^2 - y^2 = 315$, by quadratics.

Solution by WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Ohio; E. P. NORTON, Allen, Michigan; and the PROPOSER.

From (1), $x^2 - 324 = y^2 - 9$ ....(3), or $x - 18 = \frac{y^2}{x+18} - \frac{9}{x+18}$ ....(4).

From (2), $x=\frac{75}{1+y}-\frac{y}{1+y}$ .... (5), or $x-18=\frac{57}{1+y}-\frac{19y}{1+y}$ .... (6).

From (4) and (6) by reduction we have $y^2+\frac{19(x+18)}{1+y}.y=9+\frac{57(x+18)}{1+y}$ .... (7).

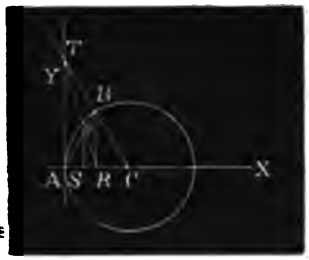
Completing the squares by adding $\left(\frac{19(x+18)}{2(1+y)}\right)^2$ to each side and reducing we find $y=3$; then (5) gives $x=18$.

Different methods were employed by Messrs. *Albert*, *Casey*, *Hart*, *Nichols*, *Robins* and *Siverly*.

103.—Proposed by WILLIAM WOOLSEY JOHNSON, Professor of Mathematics, St. Johns College, Annapolis, Maryland.

BR is an ordinata, from any point B of a circle, to the diameter passing through the fixed point A; and T is the intersection of the tangent at A with the radius produced through B. Find the locus of the intersection of AB and TR.

I.—Solution by E. A. BOWSER, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, N. J.; FRANK ALBERT Professor of Mathematics, Pennsylvania State Normal School, Millersville, Pa.; WALTER SIVERLY, Oil City, Pa.; A. R. BULLIS, Ithaca, N. Y.; and C. H. TUTTON, Wilkes Barre, Pa.



Take the diameter for the axis of X and the tangent AT for the axis of Y. Call the co-ordinates of B x' and y', and the co-ordinates of P, the intersection of AB and TR, x and y, and r the radius of the circle. Then $AT=\frac{ry'}{r-x'}$; the equation of AB is $y'=\frac{y}{x}x'$....(1); the equation of TR is $\frac{ry'}{r-x}=\frac{x'y}{x'-x}$....(2), and the equation of the circle is $y^2=(2r-x)x$...(3).

Solving (1) and (2) we obtain $x'=\frac{2rx}{x+r}$ and $y'=\frac{2ry}{x+r}$, which substituted in (3) gives us for the required locus, $y^2=rx$, a parabola, which passes through the origin A and the extremities of the vertical diameter, with parameter $=r$, and its focus one-fourth the distance from A to C.

Analagous solutions given by Messrs. *Edmunds*, *Pollard*, *Scheffer* and *Wiley*. This problem was also solved by *Lucius Brown*, the *Proposer* and Professor *Seitz*. Mr. *Brown's* solution will be given in No. 5.

104.—Proposed by W. T. R. BELL, M. A., Principal King's Mountain High School, King's Mountain, Cleaveland Co., N. C.

The railroad debt of a certain company is \$56400, interest at 7 per cent., coupons payable semi-annually. What tax must be levied to pay the interest and create such a sinking fund as will absorb the debt in 15 equal annual payments?

Solution by ROBINS FLEMING, Readington, Hunterdon County, New Jersey.

The yearly rate per cent. of \$1 at 7 per cent. interest compounded semi-annually $=(1.035)^2-1$, $=0.071225$.

Let $D=\$56400$, $r=0.07$, $R=(1+\frac{1}{2}r)^2=1.071225$, $n=15=$ number of payments and $x=$ annual payment. Then $\frac{x}{R}$, $\frac{x}{R^2}$, $\frac{x}{R^3}$, ... $\frac{x}{R^n}$ = present worths of the n payments. The sum of these present worths must be equal to the debt. $\therefore\ \frac{x}{R}+\frac{x}{R^2}+\frac{x}{R^3}+....+\frac{x}{R^n}=D$, and $x=\frac{D}{\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+....+\frac{1}{R^n}}$.

But $\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}+....+\frac{1}{R^n}=\frac{R^n-1}{R^n(R-1)}$; $\therefore\ x=\frac{DR^n(R-1)}{R^n-1}=\$6241.48$.

This problem was also solved very elegantly by Messrs. *Albert*, *Banks*, *Bullis*, *Clark*, *Seitz*, *Siverly* and *Wickersham*.

105.—Proposed by JOHN REA, Hill's Fork, Adams County, Ohio.

Find the co-ordinates a, b, a', b', of the centers, and the radii r, r' of the two circles

$$y^2+x^2-20x-40=0,\quad\text{and}\quad x^2+y^2-40y+50=0;$$

and in case the two circumferences cut each other, what are the co-ordinates of the points of intersection?
—[From *Loomis' Analytical Geometry*.

Solution by L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.

The general equations of the circles are $(x-a)^2+(y-b)^2=r^2$ and $(x-a')^2+(y-b')^2=r'^2$ respectively. Expanding and comparing with the given equations we find for the first circle $a=10$, $b=0$, $r=2\sqrt{35}=11.832+$; and for the second, $a'=0$, $b'=20$, $r'=5\sqrt{14}=18.708+$.

If the circles intersect, the values for x and y at the points of crossing must be the same for each equation.

Subtracting (1) from (2) we get $20x-40y+90=0$, $x=2y-4.5$. Substituting in equation (1), and reducing, $y=\frac{1}{10}[58\pm\sqrt{(1959)}]$, $=10.226$ or 1.374 and $x=\frac{1}{10}[71\pm2\sqrt{(1959)}]$, $\doteq15.952$ or -1.752.

NOTE.—The sign of $40y$ in the second equation was erroneously printed $+$ instead of $-$.

Solutions of the problem as printed were received from Messrs. *Bullis*, *Casey*, *Edmunds*, *Fleming*, *Heal*, *Hoover*, *Leland*, *Pollard*, *Shidy*, *Siverly* and *Tutton*.

106.—Proposed by S. C. BRACE, Philadelphia, Pennsylvania.

A circular saw, one foot in diameter, in cutting off a round log, is stopped when it reaches the center of the log, when it is found that the wood covers $\frac{1}{3}$ of the side-surface of the saw. What is the diameter of the log?

Solution by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa.; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; E. B. SEITZ, Professor of Mathematics, North Missouri State Normal School, Kirksville, Adair County, Mo.; ABRAM R. BULLIS, Ithaca, Tompkins County, New York; and W. L. HARVEY, Maxfield, Penobscot County, Maine.

Put $AC=\frac{1}{2}=a$, $BC=r$, $\angle ACB=\theta$; then $r=2a\cos\theta$, the sector BCD $=\frac{1}{2}r^2\theta=2a^2\theta\cos^2\theta$, the sector $BAC=\frac{1}{2}a^2(\pi-2\theta)$, and the triangle BAC $=\frac{1}{2}a^2\sin2\theta$; hence $DBCE=a^2(\pi-2\theta+4\theta\cos^2\theta-\sin2\theta)$,

or $\sin2\theta-2\theta\cos2\theta=\frac{2}{3}\pi$.

By Double Position we find $2\theta=125^\circ\ 46'\ 5''$ and $\theta=62^\circ\ 53'\ 2\frac{1}{2}''$.

But $r=2a\cos\theta=\cos(62^\circ\ 53'\ 2\frac{1}{2}'')=0.45579$ feet $=5.46948$ inches, and the diameter is 10.9389 inches.

Solved also by Prof. *Casey*, Sup't *Hoover*, *Geo. H. Leland*, *E. P. Norton*, Prof. *Schaffer*, *Walter Siverly* and *Chas. H. Tutton*.

107.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

At a Firemens's Fair a silver trumpet is offered to the company exhibiting the ladder that can be used in the greatest number of streets and alleys for the purpose of reaching windows on either side without changing the location of its foot. All bases and perpendiculars must be rational lengths, and a company may include in their count dimensions having as many decimal places as their ladder has, but no more.

The "Hudsons" bring a ladder 65 feet long, the "Keystones" offer one $32\frac{1}{2}$ feet in length, and the "Delawares" show one of $42\frac{1}{4}$ feet. To whom must the trumpet be awarded, and on what count?

I.—Solution by the PROPOSER.

General expressions for the sides of a right-angled triangle are $2qrs$, $q(r^2-s^2)$ and $q(r^2+s^2)$.

Hypothenuse $65=5\times13=5(3^2+2^2)=13(2^2+1^2)=1(8^2+1^2)=1(7^2+4^2)$. These values substituted in above expressions give sides of four right-angled triangles. I. 25, 60, 65; II. 39, 52, 65; III. 16, 63, 65; IV. 33, 56, 65. The "Hudsons" can use this ladder in every street whose width is the sum of any two of these 8 legs; the number of these combinations is $8\times7\div2=28$. They can also use it in every street whose width is *twice* the length of one leg—foot of ladder being in middle of street. There are 8 such streets. And they can use it in every street whose width is the length of a single leg—foot of ladder being against a house—8 streets. Count for the "Hudsons", 44 streets.

Hypothenuse $32.5=5\times5\times1.3=2.5(3^2+2^2)=6.5(2^2+1^2)=1.3(4^2+3^2)=.5(8^2+1^2)=.5(7^2+4^2)=.1(17^2+6^2)=.1(18^2+1^2)$. Substituting these values in the general expressions, we have sides of *seven* right-angled triangles. I. 12.5, 30.0, 32.5; II. 19.5, 26.0, 32.5; III. 9.1, 31.2, 32.5; IV. 8.0, 31.5, 32.5; V. 16.5, 28.0, 32.5; VI. 20.4, 25.3, 32.5; VII. 32.3, 3.6, 32.5. In all 14 legs, which taken 2 at a time give widths of $14\times13\div2=91$ streets. Ladder in middle of street, $2\times7=14$. Foot of ladder against a house, 14 streets. Count of "Keystones", 119 streets.

Hypothenuse $42.25=5\times5\times1.3\times1.3=.05(29^2+2^2)=8.45(2^2+1^2)=5(2.2^2+1.9^2)=.13(18^2+1^2)=13(1.7^2+.6^2)=3.25(3^2+2^2)=1.69(4^2+3^2)=.65(8^2+1^2)=65(.7^2+.4^2)=2.5(1.2^2+.5^2)=1(6.3^2+1.6^2)=1(5.6^2+3.3^2)$. Substituting these values as before, we have sides of *twelve* right-angled triangles. I. 41.85, 5.80, 42.25; II. 25.35, 33.80, 42.25; III. 6.15, 41.80, 42.25; IV. 41.99, 4.68, 42.25; V. 32.89, 26.52, 42.25; VI. 39.00, 16.25, 42.25; VII. 40.56, 11.83, 42.25; VIII. 40.95, 10.40, 42.25; IX. 21.45, 36.40, 42.25; X. 29.75, 30.00, 42.25; XI. 37.13, 20.16, 42.25; XII. 20.47, 36.96, 42.25; in all 24 legs, which taken two at a time give $24\times23\div2=276$ streets. Ladder in middle of street, $2\times12=24$ streets. Foot of ladder against a house, $2\times12=24$ streets. Count of "Delawares", 324 streets. Trumpet must be awarded to the "Delawares".

The solution by *Walter Siverly* is similar to this. A general solution by Dr. *Hart* will be published in No. 5.

108.—Proposed by JUSTUS F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin County, Pennsylvania.

Three army corps, A, B and C, being engaged in a campaign, have provisions sufficient for 30 weeks. On these provisions B and C would subsist 9 weeks longer than A and B; and A and C, 15 weeks longer than B and C. After 6 weeks the three army corps encounter the enemy, in consequence of which A loses $\frac{1}{8}$ of its troops, B looses $\frac{1}{6}$ and C looses $\frac{1}{4}$; also $\frac{3}{8}$ of the remaining provisions are lost. How many weeks will the remainder of the three corps subsist on the provisions left?

Solution by LUCIUS BROWN, Hudson, Middlesex County, Mass.; and ABRAM R. BULLIS, Ithaca, Tompkins County, New York.

Let unity represent the quantity of provisions at first, and x, y, z the proportions for A, B, C, respectively.

Then $x+y+z=1 \ldots\ldots (1)$, $\quad \frac{1}{y+z}-\frac{1}{x+y}=\frac{9}{30}=\frac{3}{10} \ldots\ldots (2)$, $\quad \frac{1}{x+z}-\frac{1}{y+z}=\frac{15}{30} \ldots\ldots (3)$.

Substituting in (2) and (3) the value of z in (1) gives

$$\frac{1}{1-x}-\frac{1}{x+y}=\frac{3}{10} \text{ and } \frac{1}{1-y}-\frac{1}{1-x}=\frac{1}{2}.$$

Eliminating y by comparison gives

$$\frac{10-17x-3x^2}{7+3x}=\frac{1+x}{3-x}, \text{ or } 3x^3+5x^2-71x+23=0.$$

The only value of x consistent with the conditions is $x=\frac{1}{3}$, therefore $y=\frac{1}{2}$ and $z=\frac{1}{6}$.

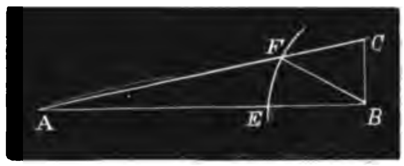
After the encounter with the enemy the provisions remaining are $\frac{5}{8}$ of $\frac{4}{5}=\frac{1}{2}$, and the quantity consumed in one week is $\frac{7}{8}$ of $\frac{1}{30}x+\frac{5}{6}$ of $\frac{1}{30}y+\frac{3}{4}$ of $\frac{1}{30}z=\frac{1}{36}$. Therefore what is left of the provisions will last the remainder of the troops 18 weeks.

Good solutions were given by Dr. *Hart*, *E. P. Norton*, *J. A. Pollard*, *L. P. Shidy*, Professor *E. B. Seitz*, *Walter Siverly* and the *Proposer*.

109.—Proposed by THOMAS BAGOT, Principal Shelby School, Canaan, Jefferson County, Indiana.

A field in the form of a right-angled triangle has a base and perpendicular of 40 and 200 feet respectively. What length of rope attached at the vertex of the right angle will permit a horse to graze upon half the field?

Solution by JULIAN A. POLLARD, Windsor, Windsor County, Vermont; ABRAM R. BULLIS, Ithaca, Tompkins County, N. Y.; and LUCIUS BROWN, Hudson, Middlesex County, Massachusetts.



Let ABC represent the field, AB = 200 feet, BC = 40 feet, and let x = BE = BF = length of rope, $\theta = \angle$ EBF.

AEF = EFCB = half the field = 2000 square feet, ABF = $100x\sin\theta$, EBF = $\frac{1}{2}x^2\theta$, BFC = $20x\cos\theta$.

$\therefore\ 100x\sin\theta-\frac{1}{2}x^2\theta=2000 \ldots (1)$, $\ 20x\cos\theta+\frac{1}{2}x^2\theta=2000 \ldots (2)$.

Adding (1) and (2) we get $x=\frac{200}{5\sin\theta+\cos\theta}$. Substituting this value of x in (2) and reducing, $10\theta=25\sin^2\theta-\cos^2\theta$, or $10\theta+13\cos 2\theta=12$, from which we find $\theta=28^\circ\ 45'\ 45''$ and $x=60.928$ feet.

Solved also by Messrs. *Albert*, *Casey*, *Harvey*, *Hoover*, *Leland*, *Shidy*, *Seitz*, *Siverly*, *Tutton* and *Wiley*.

110.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Pa.

Give an expeditious method of approximating to the square root of a quantity, and find by it the square root of 2 to at least one hundred and fifty places of decimals.

I.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

By the usual method extract the root in full to one more than half the number of figures required: then using double the root found as a divisor, and the remainder as a dividend, contract the division by cutting off one figure from the right of the divisor for each new figure of the root.

Extracting in full to 78 places I find

$\sqrt{2}=$ 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432
764157273501384623091229702492483605585073721264412149709993583 +.

There are various methods of extracting roots by approximation, but to the best of my knowledge, when used to extract the square root, in all of them, the work is more than by the above method.

II.—Solution by K. S. Putnam, Rome, Oneida County, New York.

1. Let $n=$ the number, and $x=$ its square root; then $x^2=n$. Suppose the root $>r$ and $<r+1$, and let $r+p=$ the root; p will then be a fraction, and p^2 quite small.

$$\therefore\ x^2=n=r^2+2rp+p^2;\ \text{or, rejecting } p^2,\ r^2+2rp=n,\ p=\frac{n-r^2}{2r},\ x=r+p=r+\frac{n-r^2}{2r}=\frac{n+r^2}{2r}.$$

By constantly substituting the derived value of x for r in the equation we shall approximate to the value of x.

2. If $x^2=n=2$, then $r=1$ and we have, from $x=r+p$, $p=\frac{1}{2}$; $\therefore\ x=\frac{3}{2}$. Putting $r=\frac{3}{2}$ we find for second value, $x=\frac{17}{12}$; third value, $x=\frac{577}{408}$; and we see that assuming any value of x thus derived $=\frac{a}{b}$, the next value will be $x=\frac{2a^2-1}{2ab}=\frac{a}{b}-\frac{1}{2ab}$.

We have then $\quad (1)\ x=\frac{a}{b}=\frac{3}{2},\quad (2)\ x=\frac{2a^2-1}{2ab}=\frac{17}{12},\quad (3)\ x=\frac{2(2a^2-1)^2-1}{2.2ab(2a^2-1)}=\frac{577}{408},$

$$(4)\ x=\frac{2[2(2a^2-1)^2-1]^2-1}{2.2.2.ab(2a^2-1)[2(2a^2-1)^2-1]}=\frac{665857}{470832},\qquad (5)\ x=\frac{886731088897}{627013566048}.$$

Substituting the numerator of (5) for a in (4) and the denominator for b we have

$$x=\sqrt{2}=\frac{489266466934428881954586808839856694558492182258668587145547700898547222910968507268117881704646657}{34596868615919099765318545389014861517389860071988342648187104766246565694525469768825292176881232}.$$

This fraction will give the root true to 195 or 196 decimal places, and the next correction is a fraction having 1 for its numerator and for a denominator twice the product of the numerator and denominator of the above fraction. By applying this correction the root will be found to over 390 decimals. It is quite easy to correct the root to 200 places, and I find

$\sqrt{2}=$ 1.41421356237309504880168872420969807856967187537694807317667973799073247846210703885038753432 76415727350138462309122970249248360558507372126441214970999358314132226659275055927557999505011527820 60837320076+.

If the numerator and denominator of the last-found value of x be substituted for a and b in (4) the root can be found true to over 1560 decimal places, and then quite easily corrected to 1600.

The root was not found to the required number of places in any of the other solutions received.

List of Contributors to the Junior Department.

Walter Siverly, Oil City, Venango Co., Pa., solved all the problems. Abram R. Bullis, Ithaca, Tompkins Co., N. Y., solved all but 95 and 102. Frank Albert, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster Co., Pa., solved all but 107, 108 and 110. E. B. Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Adair Co., Mo,. solved all but 95, 102, 107 and 110. Professor W. P. Casey, San Francisco, California, solved all but 103, 104, 107 and 110. L. P. Shidy, U. S. Coast Survey Office, Washington, D. C., solved all but 95, 99, 102, 103, 104 and 110. Julian A. Pollard, Windsor, Windsor County, Vermont, solved all but 95, 99, 101, 102, 104, 107 and 110. William Hoover, Superintendent of Schools and Mathematical Editor of the *Wittenberger*, Wapakoneta, Auglaize County, O., solved all but 88, 90, 95, 103, 104, 107, 108 and 110. Sylvester Robins, North Branch Depot, Somerset Co., N. J,. solved 88, 89, 90, 91, 92, 94, 96, 97, 98, 100, 102, 107 and 110. Dr. David S. Hart, M. A., Stonington, Conn., solved 88, 89, 91, 92, 94, 96, 97, 98, 100, 102, 107 and 108. Charles H. Tutton, Wilkes Barre, Pa., solved 91, 94, 96, 97, 99, 100, 101, 103, 105, 106, 109 and 110. Robins Fleming, Readington, N. J., solved 88, 89, 90, 91, 92, 94, 96, 98, 100, 104, 105 and 107. William Wiley, Detroit, Mich., solved 88, 89, 91, 92, 93, 96, 98, 100, 101, 103 and 109. J. F. W. Scheffer, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa., solved 91, 93, 94, 96, 97, 98, 101, 103, 106, 108 and 109. K. S. Putnam, Rome, Oneida County, New York, solved 88, 89, 90, 91, 92, 94, 96, 97, 98, 100 and 110. George H. Leland, Windsor, Windsor Co., Vermont, solved 88, 89, 90, 92, 96, 97, 98, 100, 105, 106 and 109. Marcus Baker, U. S. Coast Survey Office, Washington, D.C., solved 89, 91, 93, 95, 96, 97, 98, 99, 100 and 105. John I. Clark, Moran, Clinton County, Indiana, solved 88, 89, 91, 92, 93, 94, 96, 98, 100 and 104. Hon. J. H. Drummond, Portland, Maine, solved 88, 89, 91, 92, 94, 98, 100 and 110. Prof. E. J. Edmunds, B. S., Principal of Academic School No. 3, New Orleans, Louisiana, solved 89, 91, 98, 99, 100, 103, 105 and 110. David Wickersham, County Surveyor, Wilmington, O., solved 88, 89, 91, 92, 93, 94, 96 and 104. Gavin Shaw, Kemble, Ontario, Canada, solved 88, 89, 91, 92, 93, 100, 102 and 110. E. P. Norton, Allen, Michigan, solved 88, 89, 91, 96, 98, 100, 102 and 106. Professor Hugh S. Banks, Instructor in English and Classical Literature, Newburg, N. Y., solved 88, 89, 91, 92, 97, 98 and 100. Walter S. Nichols, Editor Insurance Monitor, New York, N. Y., solved 88, 89, 91, 92, 96, 98 and 102. W. L. Harvey, Maxfield, Maine, solved 92, 96, 97, 98, 100, 106 and 109. Thomas P. Stowell, Rochester, N. Y., solved 88, 89, 91, 93, 100 and 110. V. Webster Heath, Rodman, N. Y., solved 88, 89, 90, 92, 96 and 98. John S. Royer, Principal Public Schools, Ansonia, O., solved 88, 89, 92, 96, 98 and 100. Lucius Brown, Hudson, Mass., solved 97, 99, 103, 108 and 109. F. P. Matz, M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Georgia, solved 88, 89, 96, 97 and 100. O. D. Oathout, Read, Iowa, solved 89, 91, 92, 98 and 100. H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C., solved 88, 96, 97 and 99. E. A. Bowser, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, New Jersey, solved 97, 98, 100 and 103. W. E. Heal, Wheeling, Indiana, solved 96, 97, 100 and 105. Theo. L. DeLand, Office of the Secretary of the Treasury, Washington, D. C., solved 88, 89, 91 and 100. D. W. K. Martin, Webster, Darke Co., O., solved 89, 91 and 98. W. T. Magruder, Stevens Institute, Hoboken, N. J., solved 88, 89 and 100. B. F. Burleson, Oneida Castle, N. Y., solved 88, 89 and 90. J. V. Stewart, Muncie, Ind., solved 88, 89 and 98.

THOMAS BAGOT, Principal Shelby School, Canaan, Ind., solved 88, 89 and 90. W. W. JOHNSON, Professor of Mathematics, St. John's College, Annapolis, Md., solved 99 and 103. J. R. FAGAN, Erie, Pa., solved 88 and 89. D. B. O'CONNER, Union City, Ind., solved 89 and 100. G. G. WASHBURN, North East, Erie Co., Pa., solved 88 and 89. I. H. TURRELL, Cumminsville, O., solved 93. GEORGE A. JOPLIN, Danville, Kentucky, solved 89. JAMES Q. BRIGHAM, Walton, Kansas, solved 92. MRS. ANNA T. SNYDER, Chicago, Ill., solved 98. J. D. WILLIAMS, Superintendent of Schools, Sturgis, Mich., solved 102. DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, New Jersey, solved 97.

The first prize is awarded to K. S. PUTNAM, Rome, Oneida County, N. Y., and the second prize to WALTER SIVERLY, Oil City, Venango County, Pa.

PROBLEMS.

148.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A man bought a horse for $100, and sold him for $60, and then bought him back for $75. How much did he lose by the transaction?

149.—Proposed by JOHN I. CLARK, Moran, Clinton County, Indiana.

My grocer sold me a cheese, which he said weighed 32 pounds; but when it was placed on the other side of the scales, it only weighed 18 pounds. He then proposed that I should buy another of the same size, and weigh it on the opposite side from the first, to which I consented. Did I gain or lose by the transaction? And what was the true weight of the cheese?

150.—Proposed by W. S. HINKLE, Leacock, Lancaster County, Pennsylvania.

How much more can a bank make in 528 days, with $10000, by discounting notes on 45 days time, than by discounting them on 30 days, the rate of discount being 6 per cent., and the profits in both cases retained in bank till the expiration of the time?

151.—Proposed by G. F. MEAD, Uniontown, Fayette County, Pennsylvania.

If the sides of *any* quadrilateral be bisected, and the points of bisection joined, the resulting figure will be a parallelogram and equal in area to half the quadrilateral.

152.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A company of n men were counting their money. The first said to the second, "Give me your money and I will have $\$a$"; the second said to the third, "Give me $\frac{1}{2}$ of yours and I will have $\$a$"; the third said to the fourth, "Give me $\frac{1}{3}$ of yours and I will have $\$a$"; ...... the nth said to the first, "Give me one-nth of yours and I will have $\$a$". What sum had each?

153.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

What are the first ten numbers which, when multiplied by 1, 2, 3, 4, etc., will give products containing the same figures, and in the same order though beginning at a different digit, but when multiplied by 7, 17, —, —, etc., respectively, will give products containing all nines?

154.—Proposed by JOSEPH FICKLIN, M. A., Ph. D., Professor of Mathematics and Astronomy, University of the State of Missouri, Columbia, Boone County, Missouri.

The area of a triangle ABC is b; the side AB is a, and the angle opposite AB is β. Required the sides AC and BC.

155.—Proposed by G. F. MEAD, Uniontown, Fayette County, Pennsylvania.

If from any point in a diagonal of a parallelogram lines be drawn to the opposite angles the parallelogram will be divided into two pairs of equivalent triangles.

156.—Proposed by THEO. L. DELAND, Office of the Secretary of the Treasury, Washington, D. C.

A owes B $1750, payable as follows: 70 notes of $25 each, the first payable in one month, the second in two, and so on to the last which is payable in seventy months; each bears simple interest at 10 per cent. per annum payable with the note. When will he neither gain nor lose by borrowing money at 8 per cent. per annum simple interest to pay all the unpaid notes and interest, and give new notes, interest due added, payable, the first one month from the date of the change, the second two months, and the last on the date when the original notes terminated?

157.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

It is required to prove that a polynomial of n terms may be found which can be divided by a polynomial of n terms and give a quotient of n terms, the coefficient of every term in all of them being unity.

158.—Proposed by G. F. MEAD, Uniontown, Fayette County, Pennsylvania.

If two circles intersect, the common chord produced will bisect the common tangent.

159.—Proposed by WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Auglaize County, Ohio.

August, 1879, had five Fridays, Saturdays and Sundays; when will the month of August have five of each of these days again?

160.—Proposed by L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.

Find a point within a given quadrilateral, such that when joined to the middle points of the sides the quadrilateral will be divided into four equivalent parts.

161.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

What rate per cent. of interest paid in advance is equivalent to r per cent. paid at the end of the year?

162.—Proposed by O. H. MERRILL, Mannsville, Jefferson County, New York.

Two equal circles intersect, the center of each being on the circumference of the other. A circle is drawn touching that diameter of the right-hand circle which joins the centers of the given circles, and the circumferences of both circles, the right-hand one internally and the other externally; also a circle is drawn touching the one last drawn and the circumferences of both the given circles. Find the radii of these two circles.

163.—Proposed by THEO. L. DELAND, Office of the Secretary of the Treasury, Wasington, D. C.

In 1861 a 6 per cent. 20-year coin bond of the U. S., interest payable semi-annually, sold on the market for $0.891 on the dollar; what, on this basis, would have been the market value of a 4 per cent. 28-year coin bond of the U. S., interest payable quarterly?

164.—Proposed by V. WEBSTER HEATH, Rodman, Jefferson County, New York.

A field 81 yards square has one of its corners clipped off by a line meeting the sides at 60 and 80 yards respectively from the corner. A man commences plowing around the field, turning a furrow one foot wide. How many "rounds" will he plow before the unplowed portion of the field becomes a square?

165.—Proposed by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

A railroad company has $400000 of preferred stock and $300000 of common stock. The agreement is that the net income each year shall be applied to the payment of six per cent. on the preferred stock, and the balance shall be divided on the common stock. The net income is $36000 rental paid annually; a debt of $50000 is created payable in twenty years with annual interest at six per cent. (payable at the same time as the rent) in such manner that the annual interest is payable out of the current income, but the principal out of all the income after the debt becomes due until it is paid. It is agreed to create a sinking fund. What amount must be carried to the sinking fund annually, assuming that five per cent. compound interest may be earned to extinguish the debt when it becomes due, and how shall that amount be apportioned on the two kinds of stock?

166.—Proposed by Dr. S. HART WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Required the variation v of the magnetic needle, in latitude $\lambda = 42° \ 30'$, the declination of the North Star being $\delta = 88° \ 40'$, and its magnetic bearing being $b = 8° \ 48' \ 30''$ when at its greatest elongation east, and $b' = 5° \ 11' \ 30''$ when at greatest elongation west.

167.—Proposed by B. F. BURLESON, Oneida Castle, Oneida County, New York.

Given $3(x^2+y^2)=2xy(x+y)$ and $xy(x^2-y^2)=13(x-y)$, to find accurate expressions for the values of x and y.

168.—Proposed by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.

The position of a ball on a circular billiard table is given. Which path must the ball describe in order to pass through its original position after touching the cushion twice?

169.—Proposed by H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C.

Required (1) the radius of the auger that will cut out one-nth part of the surface, and (2) the radius of the auger that will cut out one-mth of the volume of a sphere, radius r, the axis of the auger coinciding with a diameter of the sphere.

170.—Proposed by K. S. PUTNAM, Rome, Oneida County, New York.

Traveling recently on a train moving 30 miles an hour and overhauling a freight moving 20 miles an hour, I endeavored to ascertain the length of the freight. One minute from the time I was opposite the rear of the freight a third train came between and put a stop to my investigation. At the next station we stopped, the door of the depot being directly opposite my seat. The freight here overhauled us, the front of the train being opposite me, when our train started. When we had attained the same rate of speed as the freight I was opposite the same point in the freight as when I closed my first obsrvation.

It took our train $1\frac{1}{2}$ minutes to get under full headway, during which time its motion was uniformly accelerated. How long was the freight train, and how far was I from the depot when I passed the freight?

171.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In a plane triangle ABC the center of the circumscribed circle is O, the center of the inscribed circle is I and the intersection of the perpendiculars is H. Knowing the sides of the triangle OIH, determine the sides of the triangle ABC.

172.—Proposed by W. W. JOHNSON, Professor of Mathematics, St. John's College, Annapolis, Maryland.

The base of a triangle is fixed and the difference between the vertical angle and one of the angles at the base is constant. Find the locus of the vertex. Discuss the curve and consider the cases in which the constant difference is 0 and 90°.

173.—Proposed by Dr. SAMUEL HART WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Required the variation v of the magnetic needle, in latitude $\lambda = 42° 30'$, the declination of the sun being $\delta = 20°$ N., and the magnetic bearing of its upper limb when rising on a horizon elevated $h = 1°$ is $b = 69° 21' 40''$. Radius of sun $= r = 16'$, and refraction $= p = 35'$.

174.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Mass.

To find by quadratic equations a triangle of which the angles are given and the distances of the vertices from a given point in the plane of the triangle.

Solutions of these problems should be received by September 1, 1880.

SENIOR DEPARTMENT.

Solutions of Problems Proposed in No. 3.

111.—Proposed by GAVIN SHAW, Kemble, Ontario, Canada.

It is required to divide a given number a so that the continued product of all its parts shall be the greatest possible.

Solution by L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; W. E. HEAL, Wheeling, Delaware County, Indiana; and WALTER S. NICHOLS, Editor *Insurance Monitor*, New York, N. Y.

Assume that the number is divided into any number of parts b, c, d, e, f, g, h, etc., the first being the greatest, so that $b+c+d+e+f+g+h+\text{etc.} = a$.

Let $c = b-p$, $d = b-q$, $e = b-r$, $f = b-s$, etc.; then $b(b-p)(b-q)(b-r)(b-s)\text{etc.}$ = a maximum. It is evident that the smaller the differences p, q, r, s, etc., become, the greater the continued product will be, for $b-(p-1)$ is greater than $b-p$, and hence the product becomes a maximum when $b=c=d=e=f=g=h=$ etc.

Having shown that the number must be divided into equal parts, let us assume that there are x such parts; then $\left(\frac{a}{x}\right)^x$ = a maximum, or $x\log\left(\frac{a}{x}\right)$ = a maximum.

Let $y = x\log\left(\frac{a}{x}\right)$. Differentiating, $\frac{dy}{dx} = \log\left(\frac{a}{x}\right) - 1$. Placing this differential coefficient $= 0$, we find $\log\left(\frac{a}{x}\right) = 1$, which shows that $\frac{a}{x}$ must be the base of the Naperian system of logarithms, or $\frac{a}{x} = 2.71828 = \varepsilon$. Therefore $x = \frac{a}{\varepsilon}$. Substituting this value of x in the second differential coefficient, the result is negative; hence $\left(\frac{a}{x}\right)^x$ is a maximum when $x = \frac{a}{\varepsilon}$.

Solved also by Messrs. *Baker*, *Bullis*, *Casey*, *Edmunds*, *Harvey*, *Hoover*, *Kummell*, *Norton*, *Oathout*, *Pollard*, *Seitz*, *Shaw*, *Siverly* and *Tutton*.

112.—Proposed by ROBINS FLEMING, Readington, Hunterdon County, New Jersey.

Find 34 right-angled triangles having the same hypothenuse.

Solution by Dr. DAVID S. HART, Stonington, New London County, Connecticut.

Let x, y be the legs and z the hypothenuse of a right-angled triangle; then $x^2+y^2=z^2$. The general values of x, y are $x=\dfrac{(p^2-q^2)z}{p^2+q^2}$, $y=\dfrac{2pqz}{p^2+q^2}$, the sum of whose squares is $=z^2$.

The formula $4t+1$ contains all the prime numbers each of which is the sum of two squares. The four least of these numbers are 5, 13, 17, 29, whose product is 32045 which put $=z$. Each of these factors is the sum of two squares in *one* way; the product of every two is the sum of two squares in *two* ways; the product of every three is the sum of two squares in *four* ways, and the product of all four is the sum of two squares in *eight* ways; so that the whole number of ways is *forty*.—See *Barlow's Theory of Numbers*. pp. 176, 177.

The following table contains, in perpendicular columns, the sum of two squares, p^2+q^2, equal successively to each of the above factors, and combinations of factors, and number of ways in which each combination may be varied; p, q the roots of two squares in each case; the values of x, y in each case, which belong to the common hypothenuse $z=32045$, in all 40 sets of right-angled triangles, any 34 of which will answer the conditions of the problem. As any four factors derived from the formula $4t+1$ may be used, the number of answers to the problem is infinite

p^2+q^2	p	q	x	y	p^2+q^2	p	q	x	y
5	2	1	19227	25636	13	3	2	12325	29580
17	4	1	28275	15080	29	5	2	23205	22100
65	7	4	16269	27608	65	8	1	31059	7888
85	7	6	4901	31668	85	9	2	29029	13572
145	12	1	31603	5304	145	9	8	3757	31824
221	14	5	24795	20300	221	11	10	3045	31900
377	16	11	11475	29920	377	19	4	29325	12920
493	22	3	30875	8580	493	18	13	10075	30420
1105	33	4	31117	7656	1105	32	9	27347	16704
"	31	12	23693	21576	"	24	23	1363	32016
1885	43	6	30821	8772	1885	42	11	27931	15708
"	38	21	17051	27132	"	34	27	7259	31212
2465	49	8	30381	10192	2465	47	16	25389	19552
"	44	23	18291	26312	"	41	28	11661	29848
6409	80	3	31955	2400	6409	75	28	24205	21000
"	72	35	19795	25200	"	60	53	3955	31800
32045	179	2	32037	716	32045	178	19	31323	6764
"	173	46	27813	15916	"	166	67	23067	22244
"	163	74	21093	24124	"	157	86	17253	27004
"	142	109	8283	30956	"	131	122	2277	31964

Solutions were also received from the *Proposer* and Messrs. *Davis*, *Drummond*, *Heal*, *Pollard*, *Robins*, *Shidy* and *Siverly*.

113.—Proposed by FREDERICK S. SAMUELS, Cerro Gordo, Inyo County, California.

What is the volume of a chip cut at an angle of 45 degrees to the center of a round log, radius r?

Solution by WILLIAM HOOVER, Superintendent of Schools and Mathematical Editor of the *Wittenberger*, Wapakoneta, Ohio; WALTER SIVERLY, Oil City, Pa.; and E. B. SEITZ, Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.

Conceive a plane to cut the chip parallel to the "crease" and perpendicular to the end of the log, at the distance x from the center. The section is a rectangle, sides $2x$, $2\sqrt{(r^2-x^2)}$ and area $4x\sqrt{(r^2-x^2)}$. The volume is expressed by

$$4\int_0^r x\,dx\sqrt{(r^2-x^2)},=\tfrac{4}{3}r^3.$$

Good solutions given by Messrs. *Albert*, *Bullis*, *Bowser*, *Hart*, *Harvey*, *McAdam*, *Norton*, *Putnam*, *Scheffer* and *Wood*.

114.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

To find two numbers such that their sum shall be a square, the sum of their squares a square, and if the cube of each be added to the square of the other the sums shall be equal.

I.—Solution by Professor ASHER B. EVANS, M. A., Principal Lockport Union School, Lockport, Niagara County, New York.

Let x and y represent the required numbers; then must

$$x+y=\square\,\ldots\ldots\ldots(1),\qquad x^2+y^2=\square\,\ldots\ldots\ldots(2),\qquad x^2+y^3=y^2+x^3\,\ldots\ldots\ldots(3).$$

Condition (3) readily reduces to $x+y=x^2+xy+y^2$, which on putting $x=ny$ gives

$$y=\frac{n+1}{n^2+n+1}\,\ldots\ldots\ldots\ldots(4).$$

On making the same substitution in (1) and (2) we obtain

$$y(n+1)=\frac{(n+1)^2}{n^2+n+1}=\square\ldots\ldots\ldots\ldots(5),\qquad y^2(n^2+1)=\square\ldots\ldots\ldots\ldots(6).$$

It now only remains to make n^2+1 and n^2+n+1 squares. The first of these expressions becomes a square when $n=\frac{r^2-1}{2r}$, and the second then becomes $n^2+n+1=\frac{r^4+2r^3+2r^2-2r+1}{4r^2}=\square$.

To make this expression a square put $r^4+2r^3+2r^2-2r+1=(r^2+r+1)^2$; this condition gives $r=-4$.

To obtain a positive value of r, put $r=p-4$ and let $(p^2-7p-13)$ be the root of

$$r^4+2r^3+2r^2-2r+1=p^4-14p^3+74p^2-178p+169;$$

then $p=\frac{360}{51}$, $r=p-4=\frac{52}{17}$, $n=\frac{2415}{1768}$; $x=\frac{101\ 1945}{13227769}$, $y=\frac{7395544}{13227769}$.

Solved also by Messrs. *Albert*, *Bullis*, *Drummond*, *Davis*, *Hart*, *Hoover* and *Siverly*. Mr. *Drummond's* solution, and some others, will be published in No. 5.

115.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

There is a series of parallelopipeds whose dimensions differ from a perfect cube by one unit in only one of its edges. In every case the solid diagonal is an integer. Calling the one whose edges are 1, 2, 2 the *first* parallelopiped, it is required to find general expressions for the dimensions of the *n*th solid, and compute the length, breadth and thickness of the 30th one.

I.—Solution by the PROPOSER.

Let x, x and $x\pm1$ be the dimensions of every solid in the series. Then $3x^2\pm2x+1=\square=(mx\pm1)^2$ $=m^2x^2\pm2mx+1$, whence $x=\frac{2m-2}{3-m^2}$ or $\frac{2-2m}{m^2-3}$. If $\sqrt{3}$ be expanded as a continued fraction the convergents are $\frac{1}{1}$, $\frac{2}{1}$, $\frac{5}{3}$, $\frac{7}{4}$, $\frac{19}{11}$, $\frac{26}{15}$, $\frac{71}{41}$, $\frac{97}{56}$, $\frac{265}{153}$, $\frac{362}{209}$, $\frac{989}{571}$, $\frac{1351}{780}$, $\frac{3691}{2131}$, $\frac{5042}{2911}$, $\frac{13775}{7953}$, $\frac{18817}{10864}$, $\frac{51409}{29681}$, $\frac{70226}{40545}$, etc. These values being substituted for m in the formula above will give the dimensions of the solids.

Using $m=2$ we get the edges of our first solid, 1, 2, 2 and solid diagonal 3. The second parallelopiped has for its edges 6, 6, 7 and for its solid diagonal 11; the third has edges 24, 24, 23 and diagonal 41; the fourth has edges 88, 88, 89 and diagonal 153; the fifth, edges and diagonal 330, 330, 329 and 571; the sixth, 1230, 1230, 1231 and 2131; the seventh, 4592, 4592, 4591 and 7953; the eighth, 17136, 17136, 17137 and 29681.

Now it must be remarked that the sum of the edges of our solids give the odd numerators in the series of convergents, 1, 5, 19, 71, 265, 989, 3691, etc., and the solid diagonals are the denominators of the same odd convergents, 1, 3, 11, 41, 153, 571, 2131, etc., etc. Here we must notice that every *even* numerator having twice its square diminished by unity will give the numerator of the convergent standing twice as high in the series, and every *odd* numerator when squared and increased by unity will give the numerator of the convergent standing twice as high in the series. So much for the numerators of the even convergents. Every *odd* numerator is the difference of the two *even* ones between which it stands, and every *odd* denominator is $\frac{1}{2}$ of the sum of the two even numerators between which it stands. By observing this rule we step from the 15th and 16th numerators to the 30th, 31st and 32d. Then by same law we reach the 61st convergent, $\frac{196736618251755451}{113585939507107651}$. From this the dimensions of 30th solid are found to be 65578872750585150, 65578872750585150 and 65578872750585151, and solid diagonal, 113585939507107651.

From the numerators of the 12th and 13th convergents we obtain those of the 25th and 26th; from these the 50th and 52d are obtained; another step and we get the 101st, $\frac{540550349649098297774205595 39}{312086889880453231135277649 71}$.

Hence the edges of the 50th solid are 180183449883032765924735198 46, 180183449883032765924735198 46 and 180183449883032765924735198 47, and the solid diagonal is 312086889880453231135277649 71.

Using the numerators of the 100th and 101st convergents we easily obtain the 201st fraction,

$$\frac{213901351831573469826566310477012163908746981296886 8432585}{123496003059983792868233973670999851237373943296493 9784153}.$$

From this fraction the dimensions of the 100th block are found to be

width,	71300450610524489942188770159004054636248993765628947752 8,
thickness,	71300450610524489942188770159004054636248993765628947752 8,
length,	71300450610524489942188770159004054636248993765628947752 9,
solid diagonal,	12349600305998379286823397367099985123737394329649397841 53.

This problem was also solved by Messrs. *Albert*, *Bullis*, *Davis*, *Heal*, *Hoover*, *Hart*, *Seitz* and *Siverly*. Mr. *Davis*' solution will be published in No. 5.

116.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and JOSEPH B. MOTT, Neosho, Mo.

Solve the equation $x^{x^x}=a$, and find the value of x when $a=300$.

I.—Solution by DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

Let $x=\varepsilon+y$, ε being the Naperian base, then taking the logarithm of the logarithms of both members we have $(\varepsilon+y)\log(\varepsilon+y)+\log[\log(\varepsilon+y)]=\log(\log a)=N.$

Developing by Maclaurin's Theorem gives

$$N=\varepsilon+\left(\frac{1+2\varepsilon}{\varepsilon}\right)y-\left(\frac{2-\varepsilon}{2\varepsilon^2}\right)y^2+\left(\frac{6-\varepsilon}{6\varepsilon^3}\right)y^3-\left(\frac{12-\varepsilon}{12\varepsilon^4}\right)y^4+\text{etc.}$$

Reverting this series gives

$$y=\left(\frac{\varepsilon}{1+2\varepsilon}\right)(N-\varepsilon)+\left(\frac{\varepsilon(2-\varepsilon)}{2(1+2\varepsilon)^3}\right)(N-\varepsilon)^2+\left(\frac{\varepsilon(6-23\varepsilon+5\varepsilon^2)}{6(1+2\varepsilon)^5}\right)(N-\varepsilon)^3+\text{etc.}$$

If $a=300$, then $x=2.3035+$.

Solutions of this problem were received from Messrs. *Albert, Baker, Casey, Kummell, Magruder, Nichols, Oathout, Shaw* and *Siverly*. Mr. *Baker's* solution will be published in No. 5.

117.—Proposed by Dr. JOEL E. HENDRICKS, M. A., Editor of the *Analyst*, Des Moines, Polk County, Iowa.

Suppose a ball to be projected, in a horizontal direction, and due west, from a point vertically above the 40th degree of north latitude; and suppose the projectile velocity to be just sufficient to arrest its motion in space which resulted from the earth's motion on its axis, so that it will descend to the earth in a vertical plane corresponding with the meridian through the point of projection. And suppose, further, that the ball, under the circumstances, shall fall to the earth in 5 seconds. How far will the ball strike the earth *north* of the 40th parallel?

Solution by G. W. HILL, Ph. D., Nautical Almanac Office, Washington, D. C.

All we have to do is to find the distance of a point on the 40th parallel from the axis which is

$$x=\frac{a\cos\varphi}{\sqrt{[1-(2c-c^2)\sin^2\varphi]}},$$

a denoting the equatorial radius of the earth, c the compression and φ the latitude.

Thence the centrifugal force at that point is $\frac{4a\pi}{T^2}\cdot\frac{\cos\varphi}{\sqrt{[1-(2c-c^2)\sin^2\varphi]}}$, where π = ratio of circumference to diameter and T = time of a sidereal rotation of the earth.

Next get the component of this in the direction of the horizontal line toward the north, which is done by multiplying by $\sin\varphi$, and we have $\frac{2\pi a}{T^2}\cdot\frac{\sin 2\varphi}{\sqrt{[1-(2c-c^2)\sin^2\varphi]}}$.

This is the force tending to deflect the falling body in the horizontal direction. If we multiply this by $\frac{1}{2}(5)^2$ we have the required answer to the problem.

Using Bessel's dimensions of the terrestrial spheroid I get for the answer 0.218286 feet.

Thus it is seen that we do not need to take into account the heterogeneity of the earth's mass, because it affects the question only in so far as it influences the value of c, which value is obtained directly from observation.

118.—Proposed by WINFIELD V. JEFFRIES, Instructor in Mathematics, Vermillion Institute, Hayesville, Ashland County, O.

How many different combinations, each composed of n letters can be formed from m letters, of which a are one letter, b are another and c are another?

Solution by WALTER SIVERLY, Oil City, Venango County, Pa; Professor J. F. W. SCHEFFER, Mercersburg, Franklin Co., Pa.; and ABRAM R. BULLIS, Ithaca, Tompkins County, New York.

Let p be the letter of which there are a, q the letter of which there are b, and r the letter of which there are c.

If we expand

$$(1+px+p^2x^2+p^3x^3+\ldots+p^ax^a)(1+qx+q^2x^2+q^3x^3+\ldots+q^bx^b)(1+rx+r^2x^2+r^3x^3+\ldots+r^cx^c),$$
$$=1+(p+q+r)x+(p^2+q^2+r^2+pq+pr+qr)x^2+\ldots+p^aq^br^cx^{a+b+c},$$

the coefficient of x contains all the combinations of the letters taken one at a time; of x^2, all the combinations of the letters taken two at a time;....of x^n, all the combinations of the letters taken n at a time.

Putting p, q, r each $=1$, the *number* of the combinations of the letters taken n at a time is the coefficient of x^n in the expansion of

$$(1+x+x^2+x^3+\ldots+x^a)(1+x+x^2+x^3+\ldots+x^b)(1+x+x^2+x^3+\ldots+x^c).$$

This expression may be written

$$\left(\frac{1-x^{a+1}}{1-x}\right)\left(\frac{1-x^{b+1}}{1-x}\right)\left(\frac{1-x^{c+1}}{1-x}\right)\ldots\ldots\left(\frac{1-x^{t+1}}{1-x}\right), =(1-x^{a+1})(1-x^{b+1})(1-x^{c+1})\ldots\ldots(1-x)^{-t},$$

when there are t different sorts of letters.

If $m=a+b+c$, the number of combinations of the m letters taken n at a time is the coefficient of x^n in the expansion of

$$(1-x^{a+1})(1-x^{b+1})(1-x^{c+1})(1-x)^{-3}.$$

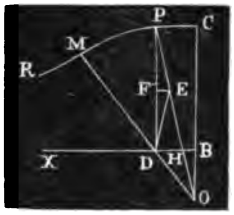
If there are $m-a-b-c=d$ separate letters the number of combinations is the coefficient of x^n in the expansion of

$$(1-x^{a+1})(1-x^{b+1})(1-x^{c+1})(1+x)^d(1-x)^{-3}.$$

119.—Proposed by W. E. Heal, Wheeling, Delaware County, Indiana.

Show how to trisect an angle, and how to construct two mean proportionals between two given straight lines, by means of the curves called the *conchoid* and the *cissoid.*

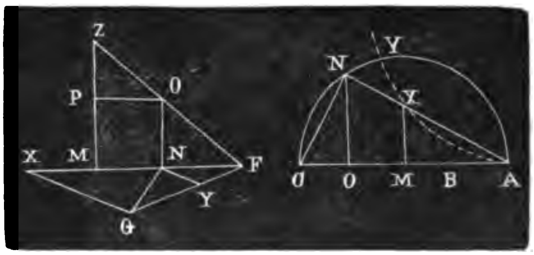
Solution, (1) by Julian A. Pollard, Windsor, Vermont; (2) and (3) by Prof. W. P. Casey, C. E., San Francisco, California.



(1). Let COM be the angle to be trisected. From any point D in one leg let fall a perpendicular DB upon the other. Take CB = 2DO, and with O as the fixed point, XB as the fixed line, construct the arc CR of the conchoid. Erect DP perpendicular to DB, and draw PO. Then is POC one-third of COM.

To prove this, bisect PH at E, and draw DE. Draw also EF parallel to DH. Since PE = EH, PF = FD, and ED = PE = EH = DO. By reason of the isosceles triangles PED and DEO, we have $\angle$DEO $=2\angle$P $=2\angle$POC. But $\angle$DEO $=\angle$EOD; therefore $2\angle$EOC $=\angle$EOD, or $\angle$EOC $=\frac{1}{3}\angle$COM.—[From *Olney's General Geometry.*

(2). Let c and d be the given extremes. Construct the rectangle MNOP, having MN $=c$ and MP $=d$. On MN construct an isosceles triangle MNG, having NG = to half MP. Make MX = MN and draw NY parallel to XG, and through G draw GF (by the conchoid) so that YF = NG, and draw FO intersecting MP produced in Z. Then NF and PZ are the required means.



For GY : YF :: XN : NF, or GY : $\frac{1}{2}d$ :: $2c$: NF; and by similar triangles ZP : c :: d : NF, $\therefore$ ZP = GY. Since MNG is isosceles, $(GF)^2 = MF\times FN + (NG)^2$. But also $(GF)^2$ = the square of the sum of ZP and half of MP, or to MZ$\times$ZP + the square of half of MP. Taking away the last from both, we get MZ$\times$ZP = MF$\times$FN, and from this and the similar triangles we have the proportions

ZM : MF :: MP : NF, ZM : MF :: NF : ZP,
ZM : MF :: ZP : MN; $\therefore$ MP : NF :: ZP : MN,
or d : MF :: ZP : c.

Any number of means may be found between two given extremes.

(3). In the right-hand figure let AB $=c$ and BC $=d$. Upon AC describe the semicircle ANC, and let AXY be the cissoid. Find the side of a square = AB$\times$BC, and let CO = side of this square. Make AM = CO, and draw ON, MX perpendicular to AC. Then from the property of the cissoid A, X and N are in a straight line, and XM and NO are the required means.

Join C and N. AM$\times$OC = MX$\times$ON $=(OC)^2=$ AB$\times$BC, therefore c : XM :: ON : d.

Mr. *Pollard* also solved the third part. *Walter Siverly* sent a good solution, and *C. H. Tutton* solved the first part.

An excellent solution of this problem may be seen in Prof. *De Volson Wood's Analytical Geometry*, pp. 218 and 220; also in *Johnson and Rice's Calculus*, pp. 284 and 286.

120.—Proposed by J. J. Sylvester, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

Prove that for all positive values of k less than unity the equation $(x+a)(x+b)=k(x+c)^2$ has two real roots.

I.—Solution by the Proposer.

Consider the sequence $(x+a)(x+b)-k(x+c)^2$; $x+a$; 1. Let x be traveled up from $-\infty$ to $+\infty$; when it reaches the vanishing point of $x+a$ the first term becoming $-k(x+c)^2$ is negative, and consequently no change in the number of continuations of sign in the sequence takes place as x passes through this point. Any change then that is brought about can only be due to x passing through a vanishing point of the first term.

Now when $x=+\infty$ there are two continuations of sign in the sequence, and when $x=-\infty$ there are none. Hence the first term must have two vanishing points; i. e. two real roots as was to be proved.

II.—Solution by WALTER S. NICHOLS, Editor *Insurance Monitor*, New York, N. Y.; DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, N. J.; and J. A. POLLARD, Windsor, Vt.

Expanding, transposing and dividing by $1-k$, we have $x^2+\frac{(a+b-2kc)x}{1-k}-\frac{kc^2-ab}{1-k}=0.$

By the Theory of Equations, this can have but two roots, and if one be imaginary so must the other. Also in order to have two imaginary roots the last term must be positive. But given $k<1$, $1-k$ is positive; also since $kx<x$, therefore $kc^2>ab$, and the last term is negative. Therefore the equation has two real roots.

Good solutions received from Messrs. *Brown*, *Bowser*, *Heal*, *Seitz*, *Siverly* and *Tutton*.

121.—Proposed by Prof. E. J. EDMUNDS, B. S., New Orleans, Louisiana.

Prove that $\Gamma\left(\frac{1}{n}\right).\Gamma\left(\frac{2}{n}\right).\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=(2\pi)^{\frac{1-n}{2}}.n^{-\frac{1}{2}}$, n being a positive integer and Γ denoting the well-known Eulerian integral.

Solution by CHARLES H. KUMMELL, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.

The *second* theorem in Γ functions,

$$\Gamma(m)\Gamma(1-m)=\int_0^{\infty}\frac{x^{m-1}dx}{1+x}=\frac{\pi}{\sin m\pi}\dots\dots(1),$$

is considered established.

Replace in (1) successively m by $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots\dots \frac{n-2}{n}, \frac{n-1}{n}$; then we have

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{n-1}{n}\right)=\frac{\pi}{\sin\frac{\pi}{n}},\ \Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{n-2}{n}\right)=\frac{\pi}{\sin\frac{2\pi}{n}},\dots\dots\Gamma\left(\frac{n-1}{n}\right)\Gamma\left(\frac{1}{n}\right)=\frac{\pi}{\sin\frac{(n-1)\pi}{n}}\dots\dots(2).$$

Taking the product of these equations we have

$$\left[\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\dots\dots\Gamma\left(\frac{n-1}{n}\right)\right]^2=\frac{\pi^{n-1}}{\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\dots\dots\sin\frac{(n-1)\pi}{n}}\dots\dots(3).$$

We have, factoring $x^{2n}-1$ by De Moivre's Theorem,

$$x^{2n}-1=(x-1)\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\dots\dots\left(x^2-2x\cos\frac{(n-1)\pi}{n}+1\right)(x+1),$$

or

$$\frac{x^{2n}-1}{x^2-1}=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\dots\dots\left(x^2-2x\cos\frac{(n-1)\pi}{n}+1\right)\dots\dots(4).$$

In this equation let x be replaced by $+1$ and -1 and it becomes

$$n=2^{n-1}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{2\pi}{n}\right)\dots\dots\left(1-\cos\frac{(n-1)\pi}{n}\right)\dots\dots(5),$$

$$n=2^{n-1}\left(1+\cos\frac{\pi}{n}\right)\left(1+\cos\frac{2\pi}{n}\right)\dots\dots\left(1+\cos\frac{(n-1)\pi}{n}\right)\dots\dots(5').$$

Multiplying (5) and (5′) together and extracting the square root of the product, we have

$$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\dots\dots\sin\frac{n-1}{n}\dots\dots(6).$$

Extracting the square root of the product of (3) and (6) we obtain finally

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\dots\Gamma\left(\frac{n-1}{n}\right)=n^{-\frac{1}{2}}(2\pi)^{\frac{n-1}{2}},$$

which is only a particular case of the more general theorem known as Gauss' third theorem in Γ functions, viz;

$$\Gamma\left(x\right)\Gamma\left(x+\frac{1}{t}\right)\Gamma\left(x+\frac{2}{t}\right)\dots\dots\Gamma\left(x+\frac{t-1}{t}\right)=(2\pi)^{\frac{t-1}{2}}t^{\frac{1}{2}-tx}\Gamma\left(tx\right).$$

The above solution is condensed from *Price's Infinitesimal Calculus*, vol. i, pp. 164 and 165.

Solved also by Messrs. *Brown*, *Bowser*, *Bullis*, *Heal*, *Hoover*, *Matz*, *Seitz*, *Siverly*, *Tutton* and the *Proposer*.

122.—Proposed by Prof. DAVID TROWBRIDGE, Waterburg, Tompkins County, New York.

When the earth is in perihelion, suppose the sun's mass to be suddenly increased by a half of itself, or so that m' becomes $\frac{3}{2}m'$. Required the change in the elements of the terrestrial orbit.

Solution by the PROPOSER.

Let m be the mass of the earth and sun together, and for convenience of calculation let m become $\frac{3}{2}m$.

Put the present mean distance of the earth equal to 1, and what it becomes when m becomes $\frac{3}{4}m$ equal to a'. Let the perihelion distance of the earth be $\rho=0.9832249$. Put V for the velocity of the earth at perihelion, which, as we shall see, will become the aphelion velocity, and ρ the aphelion distance, $T=365.2563744$ days, the earth's sidereal period before the change, and T' that after.

Works on Analytical Mechanics will give the following equations:

$$V^2=\frac{2m}{\rho}-m=\frac{3m}{\rho}-\frac{\frac{3}{4}m}{a'}\ldots\ldots(1),\quad \rho=a'(1+e')\ldots\ldots(2),\ e'\text{ being the eccentricity required,}$$

$$m=\frac{4\pi^2}{T^2}\ldots\ldots\ldots\ldots(3),\qquad \tfrac{3}{4}m=\frac{4\pi^2a'^3}{T'^2}\ldots\ldots\ldots\ldots(4).\qquad \therefore\ T'^2=T^2\times\tfrac{4}{3}a'^3\ldots\ldots\ldots\ldots(5).$$

From (1), $a'=\dfrac{3\rho}{2(1+\rho)}=0.7436$. From (2), $e'=\dfrac{\rho}{a'}-1=\dfrac{2\rho-1}{3}=0.3221$. $\dfrac{T'}{T}=\dfrac{3}{2}\sqrt{\left[\left(\dfrac{\rho}{1+\rho}\right)^3\right]}$.

$\therefore\ T'=191.2545$ days.

The position of the perihelion will be changed 180°, and the perihelion distance $=0.5040$.

Solved also by Prof. *D. J. McAdam*, Prof. *E. B. Seitz*, *Walter Siverly* and Prof. *De Volson Wood*.

123.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

To find three whole numbers such that the sum of the squares of any two of them increased by the product of the same two shall be a rational square.

I.—Solution by Rev. U. JESSE KNISELY, Ph. D., Newcomerstown, Tuscarawas County, Ohio.

Let x, y and z denote the required numbers. Then we must make

$$x^2+xy+y^2=\square=a^2\ldots\ldots(1),\quad x^2+xz+z^2=\square=b^2\ldots\ldots(2),\quad y^2+yz+z^2=\square=c^2\ldots\ldots(3).$$

Multiplying (1) by 4 times (2),

$$4(x^2+xy+y^2)(x^2+xz+z^2)=4a^2b^2c^2\ldots\ldots\ldots\ldots(4).$$

Subtracting (3) from the sum of (1) and (2),

$$2x^2+xy+xz-yz=a^2+b^2-c^2\ldots\ldots\ldots\ldots(5).$$

Subtracting square of (5) from (4),

$$3x^2y^2+6x^2yz+3x^2z^2+6xyz^2+6xy^2z+3y^2z^2=4a^2b^2-(a^2+b^2-c^2)^2.$$

Taking square root,

$$xy+xz+yz=\tfrac{1}{3}\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}=\tfrac{1}{3}A\ldots\ldots\ldots\ldots(6).$$

Adding (1), (2) and (3), transposing and dividing by 2, we have

$$x^2+y^2+z^2=\tfrac{1}{2}(a^2+b^2+c^2)-\tfrac{1}{2}(xy+xz+yz),$$

$$=\tfrac{1}{2}(a^2+b^2+c^2)-\tfrac{1}{6}\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2},$$

by (6)

$$=\tfrac{1}{2}(a^2+b^2+c^2)-\tfrac{1}{6}A\ldots\ldots\ldots\ldots(7).$$

Adding twice (6) to (7), and extracting the square root,

$$x+y+z=\tfrac{1}{2}\sqrt{2(a^2+b^2+c^2)\pm2\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}},\ =\tfrac{1}{2}\sqrt{2(a^2+b^2+c^2)\pm2A},\ =\tfrac{1}{2}B\ldots(8).$$

From (6), $x=\dfrac{\frac{1}{3}A-yz}{y+z}$, and from (8), $x=\frac{1}{2}B-(y+z)$. Hence we have $\frac{1}{2}B-(y+z)=\dfrac{\frac{1}{3}A-yz}{y+z}$.

Clearing of fractions, transposing, and reducing,

$$\tfrac{1}{2}B(y+z)=\tfrac{1}{3}A+y^2+yz+z^2,\ =\tfrac{1}{3}A+c^2\text{ by (1); whence }y+z=\frac{2A+6c^2}{3B}\ldots\ldots\ldots\ldots(9).$$

Subtracting (9) from (8),

$$x=\tfrac{1}{2}B-\frac{2A+6c^2}{3B}=\frac{3B^2-4A-12c^2}{6B}=\frac{a^2+b^2-c^2\pm\frac{1}{3}\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}{\sqrt{2(a^2+b^2+c^2)\pm2\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}}.$$

In like manner, find $x+z$ and $x+y$, and then subtracting from (8) we get

$$y=\frac{a^2+c^2-b^2\pm\frac{1}{3}\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}{\sqrt{2(a^2+b^2+c^2)\pm2\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}},\quad z=\frac{b^2+c^2-a^2\pm\frac{1}{3}\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}{\sqrt{2(a^2+b^2+c^2)\pm2\sqrt{12a^2b^2-3(a^2+b^2-c^2)^2}}}.$$

We must now make

$$12a^2b^2-3(a^2+b^2-c^2)^2=\square=A^2\ldots\ldots(10),\text{ and }2(a^2+b^2+c^2)\pm2A=\square=B^2\ldots\ldots(11).$$

Put $a=(1-n)b$, $c=(1+n)b$; then (10) becomes $A^2=(9-36n^2)b^4=\square\ldots\ldots(12)$. Therefore we must satisfy $9-36n^2=\square$, or, dividing by 9, $1-4n^2=\square$, which put $=(1-pn)^2$ and we get

$$n=\frac{2p}{p^2+4},\qquad\text{and}\qquad A=\left(\frac{3p^2-12}{p^2+4}\right)b^2.$$

Substituting in (11), and taking the upper sign,

$$6b^2+\frac{16pb^2}{(p+4)^2}+\frac{6(p^2-4)b^2}{p^2+4}=\square, \text{ or } \frac{12p^4+64p^2}{(p^2+4)^2}=\square,$$

or $3p^2+16=\square$, which put $=(4+qp)^2$ and we find $p=\frac{8q}{3-q^2}$.

The problem is now resolved, if we find *such* values of p, q, b, as will make the trinomials positive. Since we have treated c as the largest of the three absolute-term roots, we have to make $a^2+b^2>c^2$, or $2-2n+n^2>1+2n+n^2$, or $1>4n$, and $n<\frac{1}{4}$. Hence $\frac{2p}{p^2+4}<\frac{1}{4}$, $8p<p^2+4$, or $p^2-8p+16>12$, $p-4>2\sqrt{3}$ and $p>4+2\sqrt{3}$, or $>7.46+$. Hence q may be taken $=-2$, and then $p=16$, $n=\frac{8}{65}$, $a=\frac{57}{65}b$, $c=\frac{73}{65}b$. $A=\frac{189}{65}b^2$, $B=\frac{224}{65}b$, $x=\frac{195}{455}b$, $y=\frac{264}{455}b$, $z=\frac{325}{455}b$. Now take $b=455$, and we have $x=195$, $y=264$, $z=325$, which are the least values that have yet been found.

This problem was also very elegantly solved by *Reuben Davis* and Dr. *Hart*; their solutions will be published in No. 5.

124.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

The first of two casks contained a gallons of wine, and the second b gallons of water; c gallons were drawn from the second cask, and then c gallons were drawn from the first cask and poured into the second, and the deficiency in the first supplied with c gallons of water; c gallons were then drawn from the first cask, and c gallons drawn from the second and poured into the first and the deficiency in the second cask supplied with c gallons of wine. Required the quantity of wine in each cask after n such operations as that described above.

Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let u_n and v_n represent the wine in the first and second casks respectively at the end of the nth operation; the quantities of wine in each cask at the successive stages of the $(n+1)$th operation are

$$u_n,\ v_n;\ \left(1-\frac{c}{a}\right)u_n,\ \left(1-\frac{c}{b}\right)v_n;\ \left(1-\frac{c}{a}\right)^2u_n,\ \left(1-\frac{c}{b}\right)v_n+\frac{c}{b}u_n;$$

$$\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}\right]u_n+\frac{c}{b}\left(1-\frac{c}{b}\right)v_n,\ \left(1-\frac{c}{b}\right)^2v_n+\frac{c}{a}\left(1-\frac{c}{b}\right)u_n+c.$$

Whence

$$u_{n+1}=\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}\right]u_n+\frac{c}{b}\left(1-\frac{c}{b}\right)v_n\ldots.(1),\quad v_{n+1}=\left(1-\frac{c}{b}\right)^2v_n+\frac{c}{a}\left(1-\frac{c}{b}\right)u_n+c\ldots.(2).$$

Also,

$$u_{n+2}=\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}\right]u_{n+1}+\frac{c}{b}\left(1-\frac{c}{b}\right)v_{n+1}\ldots(3),\quad v_{n+2}=\left(1-\frac{c}{b}\right)^2v_{n+1}+\frac{c}{a}\left(1-\frac{c}{b}\right)u_{n+1}+c\ldots(4).$$

Eliminating v_n from (1) and (2),

$$\left(1-\frac{c}{b}\right)u_{n+1}-\frac{c}{b}v_{n+1}=\left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2u_n-\frac{c^2}{b}\ldots\ldots(5).$$

Eliminating v_{n+1} from (3) and (5),

$$u_{n+2}-\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}+\left(1-\frac{c}{b}\right)^2\right]u_{n+1}+\left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2u_n=\frac{c^2}{b}\left(1-\frac{c}{b}\right)\ldots\ldots(6).$$

Let r_1, r_2 be the roots of the equation

$$y^2-\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}+\left(1-\frac{c}{b}\right)^2\right]y+\left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2=0.$$

The solution of (6) is (*Hymer's Finite Differences*, pp. 54, 55)

$$u_n=C_1(r_1)^n+C_2(r_2)^n+\left\{\frac{\frac{c^2}{b}\left(1-\frac{c}{b}\right)}{1-\left(1-\frac{c}{a}\right)^2-\frac{c^2}{ab}-\left(1-\frac{c}{b}\right)^2+\left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2}\right\}.$$

Put the last expression $=S$. From (1), $u_0=a$, $u_1=\frac{(a-c)^2}{a}-\frac{c^2}{b}$. $C_1+C_2=a-S$,

$$r_1C_1+r_2C_2=u_1-S;\text{ whence } C_1=\frac{u_1-S-r_2(a-S)}{r_1-r_2},\quad C_2=\frac{r_1(a-S)-u_1+S}{r_1-r_2}.$$

$$u_n=\left(\frac{u_1-S-r_2(a-S)}{r_1-r_2}\right)(r_1)^n+\left(\frac{r_1(a-S)-u_1+S}{r_1-r_2}\right)(r_2)^n+S.$$

Eliminating u_n from (1), (2), and u_{n+1} from this and (4),

$$v_{n+2}-\left[\left(1-\frac{c}{a}\right)^2+\frac{c^2}{ab}+\left(1-\frac{c}{b}\right)^2\right]v_{n+1}+\left(1-\frac{c}{a}\right)^2\left(1-\frac{c}{b}\right)^2v_n=c\left[1-\left(1-\frac{c}{a}\right)^2-\frac{c^2}{ab}\right].$$

Whence $v_n = C_3(r_1)^n + C_4(r_2)^n + S_1$. From (2), $v_0 = 0$, $v_1 = c\left(2 - \frac{c}{b}\right)$;

$$\therefore v_n = \left(\frac{v_1 + (r_2 - 1)S_1}{r_1 - r_2}\right)(r_1)^n + \left(\frac{S_1(1 - r_1) - v_1}{r_1 - r_2}\right)(r_2)^n + S_1.$$

Solved also, in an elegant manner, by Professor *E. B. Seitz*.

125.—Proposed by ORLANDO D. OATHOUT, Read, Clayton County, Iowa.

What is the average thickness of a slab sawed at random from a round log?

I.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Let the annexed diagram represent a cross-section of the log, ABC being that of the slab.

The average thickness of the slab is the mean value of the ordinate DE.

Let OL = EF = w, FO = EL = x, DE = y, Δ = average thickness of the slab. Then $(w+y)^2 + x^2 = r^2$, r being the radius of the log, whence $y = \sqrt{(r^2 - x^2)} - w$.

The limits of x are 0 and $\sqrt{(r^2 - x^2)} = x'$; of w, 0 and r.

$$\therefore \Delta = \frac{\int_0^r \int_0^{x'} y\,dx\,dw}{\int_0^r \int_0^{x'} dx\,dw} = \frac{\int_0^r \int_0^{x'} [\sqrt{(r^2 - x^2)} - w]\,dx\,dw}{\int_0^r \int_0^{x'} dx\,dw} = \frac{2}{\pi r^2}\int_0^r \left[r^2 \cos^{-1}\left(\frac{w}{r}\right) - w\sqrt{(r^2 - w^2)}\right] dw = \frac{4r}{3\pi}.$$

A solution by *Walter Siverly* will be published in No. 5.

126.—Proposed by FRANCIS M. PRIEST, Bryan, Williams County, Ohio.

Divide unity into three such positive parts that if unity be added to each part the three sums shall be rational cubes.

Solution by REUBEN DAVIS, Bradford, Stark County, Illinois.

Let $\left(\frac{m+n}{z}\right)^3 - 1$, $\left(\frac{m-n}{z}\right)^3 - 1$ and $\left(\frac{y}{z}\right)^3 - 1$ represent the required parts.

Then will $\left(\frac{m+n}{z}\right)^3 - 1 + \left(\frac{m-n}{z}\right)^3 - 1 + \left(\frac{y}{z}\right)^3 - 1 = 1$ ...(1), or $(m+n)^3 + (m-n)^3 + y^3 = 4z^3$ ...(2),

or $2m^3 + 6mn^2 + y^3 = 4z^3$; therefore $n^2 = \frac{4z^3 - 2m^3 - y^3}{6m} = \frac{24mz^3 - 12m^4 - 6my^3}{36m^2}$ ....(3).

To make the numerator of (3) a square we must find such values of $m+n$, $m-n$ and y as will fulfill the condition (2). If $m = 18$, $n = 1$, $y = 12$, (2) is satisfied, z being 15; but the required parts will not all be positive. To find positive values, make the numerator of (3) a square after substituting $18 - px$ for m, $12 - qx$ for y and 15 for z.

Making these substitutions we have

$$(108)^2 + (209304p + 46656q)x - (23328p^2 + 2592pq + 3888q^2)x^2$$
$$+ (864p^3 + 216pq^2 + 108q^3)x^3 - (12p^4 + 6pq^3)x^4 = \square.$$

Assume $\left[108 + (969p + 216q)x - \frac{(106921p^2 + 46800pq + 5616q^2)x^2}{24}\right]$ for the root of this square, and after squaring and reducing we find

$$x = \frac{144(34538939p^3 + 22814712p^2q + 5184432pq^2 + 404784q^3)}{11432107153p^4 + 10007805600p^3q + 3391176672p^2q^2 + 525661056pq^3 + 31539456q^4}.$$

If we take $p = 1$ and $q = -\frac{41}{12}$, $x = \frac{23159322}{26354245}$, $m = \frac{902434176}{52708490}$, $y = \frac{790757247}{52708490}$, $n = \frac{354087676605}{52708490 \times 73584}$,

and the required parts of unity are

$$\left(\frac{m+n}{z}\right)^3 - 1 = \frac{1452917906705319052069192489567835482760 49}{196909050430895125339325091034151424000000},$$

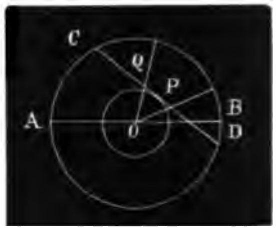
$$\left(\frac{m-n}{z}\right)^3 - 1 = \frac{51520189639391567352035150176008549814959}{196909050430895125339325091034151424000000},$$

$$\left(\frac{y}{z}\right)^3 - 1 = \frac{97070120971652780370691901359325908992}{196909050430895125339325091034151424000000}.$$

127.—Proposed by Samuel Roberts, M. A., F. R. S., Member of the London Mathematical Society, London, England.

Two random points being taken within a circle (1) on opposite sides of a given diameter, (2) on the same side, (3) anywhere; find in each case the average radius of the concentric circle touched by the chord through them.

I.—Solution by the Proposer.



Let P, Q be the two random points anywhere in the surface of the given circle whose center is O and radius r. Draw the diameter AOB and let OP $= x$, OQ $= y$, $\angle$POQ $= \omega$, $\angle$POB $= \theta$. Then if R is the required average radius,

$$R = \frac{\int_0^\pi \int_0^r \int_0^x \int_0^\pi \frac{xy\sin\omega}{\sqrt{(x^2+y^2-2xy\cos\omega)}}\, d\theta\, xdx\, ydy\, d\omega}{\int_0^\pi \int_0^r \int_0^x \int_0^\pi d\theta\, xdx\, ydy\, d\omega}$$

$$= \frac{8}{\pi^2 r^4}\int_0^\pi \int_0^r \int_0^x 2y d\theta\, xdx\, ydy = \frac{16r}{15\pi}.$$

Let P and Q be both taken in the semicircle above AOB. Then the required average radius

$$R_1 = \frac{\int_0^\pi \int_0^r \int_0^x \int_0^{\pi-\theta} \frac{xy\sin\omega}{\sqrt{(x^2+y^2-2xy\cos\omega)}}\, d\theta\, xdx\, ydy\, d\omega}{\int_0^\pi \int_0^r \int_0^x \int_0^{\pi-\theta} d\theta\, xdx\, ydy\, d\omega}$$

$$= \frac{16}{\pi^2 r^4}\int_0^\pi \int_0^r \int_0^x [\sqrt{(x^2+y^2-2xy\cos\omega)} - (x-y)]\, d\theta\, xdx\, ydy = \frac{256r}{45\pi^2} - \frac{8r}{15\pi}.$$

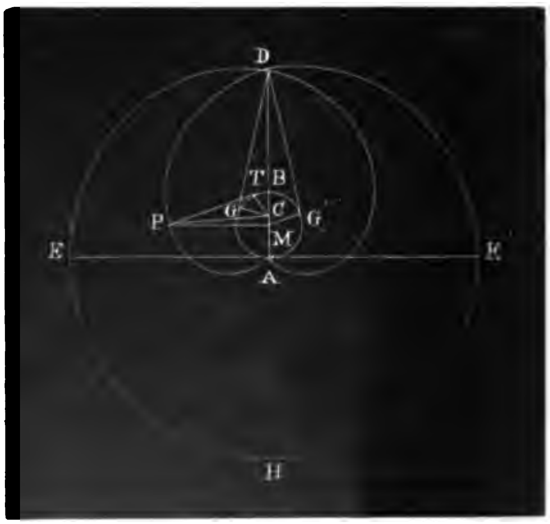
Let P be taken in the lower semicircle and Q in the upper, then $R_2 = 2R - R_1 = \frac{8r}{3\pi} - \frac{256r}{45\pi^2}$.

Elegant solutions received from Professor *Seitz* and *Walter Siverly*, which will be published in No. 5.

128.—Proposed by E. P. Norton, Allen, Hillsdale County, Michigan.

There is a circular fish pond surrounded by palisades, to the outside of which a horse is tethered. The length of the tether is equal to the circumference of the pond. Required the diameter of the pond, supposing the horse to have the liberty of grazing an acre of grass.

Solution by Charles H. Kummell, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.



Let AC $=$ BC $= a =$ radius of the pond and A the point where the tether is fastened. Suppose the horse winds his tether around the entire circumference of the pond; he will then be at A. If he unwinds the rope, keeping it stretched, he will describe an involute to the pond APDE′. From E′ he will describe a semicircle, radius AE′ $=$ AE $= 2a\pi$. From E over D to A he will again move in an involute.

The entire area in reach of the horse may be divided into three heterogeneous parts, viz: 1, the semicircle E′HE; 2, the two involute spaces AGDEA and AG′DE′A, which are equal by symmetry; 3, the space BGDG′B.

Let C be the origin, AD the x-axis and P any point; then we have

$$\text{CM} = x = a\cos\varphi + a\varphi\sin\varphi \;.......\; (1),$$

$$\text{PM} = y = a\sin\varphi - a\varphi\cos\varphi \;.......\; (2),$$

where φ denotes the angle which the radius CT, perpendicular to the radius of curvature PT, makes with the x-axis.

At the point D where the involutes intersect on the x-axis we have

CD $= x_0 = a\cos\varphi_0 + a\varphi_0\sin\varphi_0$, where $\varphi_0 =$ AGTBG′; $0 = a\sin\varphi_0 - a\varphi_0\cos\varphi_0$, $\therefore\ \varphi_0 = \tan\varphi_0 \ldots$ (3),

and $x_0 = \frac{a}{\cos\varphi_0} = a\sqrt{(1+\varphi_0^2)} \ldots\ldots$ (4). We have then BGDG′B $= a^2\varphi_0 - a^2(\varphi_0 - \pi) = a^2\pi \ldots\ldots$ (5).

Because any radius of curvature $\rho = a\varphi$ and the area element between two consecutive radii of curvature $dA = \frac{1}{2}\rho^2 d\varphi = \frac{1}{2}a^2\varphi^2 d\varphi \ldots\ldots$ (6), we have the two involute areas

$\text{AGDEA} + \text{AG}'\text{DE}'\text{A} = a^2\int_{\phi_0}^{2\pi}\varphi^2 d\varphi = \frac{1}{3}a^2(8\pi^3 - \varphi_0^3)\ldots(7)$. Finally the semicircle $\text{E}'\text{HE} = 2a^2\pi^4\ldots(8)$. Hence adding (5), (7) and (8) 1 acre = 160 square rods = 43560 square feet $= a^2(\pi + \frac{8}{3}\pi^3 - \frac{1}{3}\varphi_0^3 + 2\pi^4)$

$$= a^2(\pi + \tfrac{14}{3}\pi^3 - \tfrac{1}{3}\varphi_0^3)\ldots\ldots\ldots\ldots(9). \quad \therefore \quad a = \sqrt{\left(\frac{43560}{\pi + \frac{14}{3}\pi^3 - \frac{1}{3}\varphi_0^3}\right)}\ldots\ldots\ldots\ldots(10).$$

Solving equation (3) we obtain $\varphi_0 = \tan\varphi_0 = 4.494039 = 264° \, 27' \, 18.35''$. Using this value in (10) we obtain $a = 19.24738$ feet $= 1.1665$ rods.

Solved also by Messrs. *Brown*, *Bullis*, *Casey*, *Harvey*, *Hoover*, *McAdam*, *Seitz*, *Siverly* and *Tutton*.

129.—Proposed by Reuben Davis, Bradford, Stark County, Illinois.

It is required to find three positive numbers, such that if each be diminished by the cube of their sum the three remainders will be rational cubes.

I.—Solution by Dr. David S. Hart, M. A., Stonington, New London County, Connecticut.

Let x, y, z be the numbers, and s = their sum. Then $x - s^3 = m^3$, $y - s^3 = n^3$, $z - s^3 = p^3$; whence we have $x = m^3 + s^3$, $y = n^3 + s^3$, $z = p^3 + s^3$; and by addition, $x + y + z = m^3 + n^3 + p^3 + 3s^3 = s$; $\therefore \; m^3 + n^3 + p^3 = s - s^3$. In this expression s must be taken such a fraction that s may be greater than $3s^3$. Let $s = \frac{1}{2}$, then $s - 3s^3 = \frac{1}{8} = \frac{216}{1728} = \frac{27}{1728} + \frac{64}{1728} + \frac{125}{1728}$, which let $= m^3$, n^3, p^3, respectively; whence $x = \frac{243}{1728}$, $y = \frac{280}{1728}$, $z = \frac{341}{1728}$.

Let $s = \frac{1}{7}$, then $s - 3s^3 = \frac{46}{343} = \frac{368}{2744} = \frac{27}{2744} + \frac{125}{2744} + \frac{216}{2744}$, which let $= m^3$, n^3, p^3, respectively; whence $x = \frac{5}{392}$, $y = \frac{19}{392}$, $z = \frac{32}{392}$.

Proceeding in the same manner for four numbers, we have $w = \frac{28}{5832}$, $x = \frac{35}{5832}$, $y = \frac{370}{5832}$, $z = \frac{539}{5832}$.

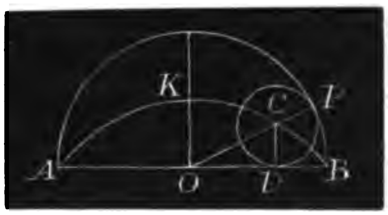
In like manner we can find 5, 6, 7, ... n numbers.

A general solution by *Reuben Davis* will be published in No. 5.

130.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A circle is inscribed at random in a given semicircle. Find (1) the average area of the circle and (2) the chance that the circle does not exceed $\frac{1}{8}$ of the semicircle.

Solution by the Proposer.



1. Let O be the middle of the base of the semicircle, and C the center of the inscribed circle. Through C draw the radius OP, and draw CD perpendicular to AB.

Let r = radius of the semicircle, OD $= x$, CD $= y$ = radius of inscribed circle and $\varDelta$ = average area required. Then OC $= r - y$ and we have $x^2 + y^2 = (r - y)^2$, or $x^2 = r^2 - 2ry$, the equation to the locus of C, which is, therefore, a parabola, axis vertical, passing through A and B, having its focus at O. The centers of the inscribed circles are uniformly distributed on this parabola, and the *number* of circles is, therefore, proportional to the length of the parabolic arc AKB.

Let s = any portion of the parabolic arc, measured from the vertex, then the average area sought is

$$\varDelta = \frac{\int \pi y^2 ds}{\int ds} = \frac{\int \pi\left(\frac{r^2 - x^2}{2r}\right)^2 ds}{\int ds} = \frac{\frac{\pi}{4r^3}\int_0^r (r^2 - x^2)^2\sqrt{(r^2 + x^2)}dx}{\int ds}, \text{ since } ds = \sqrt{(dx^2 + dy^2)} = \frac{dx\sqrt{(r^2 + x^2)}}{r};$$

$$= \pi r^2\left(\frac{13}{32} - \frac{11\sqrt{2}}{24[\sqrt{2} + \log(1 + \sqrt{2})]}\right).$$

2. Put $\frac{1}{8} = \frac{1}{n}$ and we have $\frac{\pi(r^2 - x^2)^2}{4r^2} = \frac{\pi r^2}{2n}$, from which $x = r\sqrt{\left(1 - \frac{\sqrt{2}}{\sqrt{n}}\right)} = x'$. If x is greater than x' the inscribed circle will not exceed one-nth of the semicircle.

$$\therefore p = 1 - \frac{\int_0^{x'} \sqrt{(r^2 + x^2)}dx}{\int_0^r \sqrt{(r^2 + x^2)}dx} = 1 - \frac{\left(1 - \frac{\sqrt{2}}{\sqrt{n}}\right)^{\frac{1}{2}}\left(2 - \frac{\sqrt{2}}{\sqrt{n}}\right)^{\frac{1}{2}} + \log\left[\left(1 - \frac{\sqrt{2}}{\sqrt{n}}\right)^{\frac{1}{2}} + \left(2 - \frac{\sqrt{2}}{\sqrt{n}}\right)^{\frac{1}{2}}\right]}{\sqrt{2} + \log(1 + \sqrt{2})},$$

$$= 1 - \frac{1}{2}\left(\frac{\sqrt{3} + \log(2 + \sqrt{3})}{\sqrt{2} + \log(1 + \sqrt{2})}\right).$$

Solved also very elegantly by *A. B. Bullis*, Professor *H. T. J. Ludwig*, Professor *E. B. Seitz* and *Walter Siverly*.

131.—Proposed by DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

A spherical homogeneous mass m, radius r, contracts by the mutual attraction of its particles to a radius nr; if the work thus expended be suddenly changed into heat, how many degrees F. will the temperature of the mass be increased, its specific heat being s and the heat uniformly disseminated?

I.—Solution by Abram R. Bullis, Ithaca, Tompkins County, New York.

Let $W=$ the work expended, $M=$ the earth's mass and $R=$ the earth's radius.

Then $$W=\frac{3R^2m^2}{Mr^6}\int_0^r\int_{ny}^y y^5x^{-2}dydx=\frac{3R^2m^2(1-n)}{Mnr^6}\int_0^r y^4dy=\frac{3R^2m^2(1-n)}{5Mnr}.$$

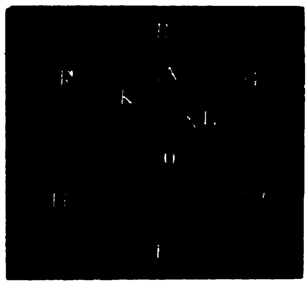
If the foot-pound be taken as the unit of work, then $H=\frac{W}{772},=\frac{3R^2m^2(1-n)}{3860Mnr}$, where H is the heat-equivalent of the work, its unit being the pound-degree Fahrenheit, and the required increase of temperature $=\frac{H}{ms}=\frac{3R^2m(1-n)}{3860Mnrs}$.

An elaborate solution by the *Proposer* will be published in No. 5.

132.—Proposed by Franklin P. Matz, M. E., M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Carroll County, Georgia.

Three persons, A, B and C, are banished to a level circular island, diameter $2r$ feet. At the center of the island is a cylindric fort, diameter $2a$ feet. During a dark and foggy night B and C stray away from A, and from each other, and lie down to rest. At the first dawn of clear morning, while A and B are yet reposing, A looks around for them. Required the probability that A, without moving from his place of observation, can see both of his companions.

Solution by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.



A can see his companions if they are anywhere in the area EHKPLJE. Let $x=$ AO, $u=$ area EHKPLJE and $p=$ the required chance.

Then $$p=\frac{\int_a^r u^2.2\pi xdx}{\int_a^r(r^2-a^2)^2\pi^2.2\pi xdx},=\frac{1}{(r^2-a^2)^3\pi^2}\int_a^r 2u^2xdx\,\ldots\ldots(1).$$

$\angle\text{AOL}=\cos^{-1}\left(\frac{a}{x}\right)$, $\angle\text{LOJ}=\cos^{-1}\left(\frac{a}{r}\right)$, $\text{JL}=\sqrt{(r^2-a^2)}$, area FKPLGE $=(r^2-a^2)\cos^{-1}\left(\frac{a}{x}\right)$; segment GLJ $=\frac{1}{2}r^2\cos^{-1}\left(\frac{a}{r}\right)-\frac{1}{2}a\sqrt{(r^2-a^2)}$, and $u=(r^2-a^2)\cos^{-1}\left(\frac{a}{x}\right)+r^2\cos^{-1}\left(\frac{a}{r}\right)-a\sqrt{(r^2-a^2)}$.

$$\therefore\ p=\frac{1}{(r^2-a^2)^3\pi^2}\int_a^r 2\left[(r^2-a^2)\cos^{-1}\left(\frac{a}{x}\right)+r^2\cos^{-1}\left(\frac{a}{r}\right)-a\sqrt{(r^2-a^2)}\right]^2xdx\,\ldots\ldots(2).$$

$$\int 2xdx\left\{\cos^{-1}\left(\frac{a}{x}\right)\right\}^2=x^2\left\{\cos^{-1}\left(\frac{a}{x}\right)\right\}^2-\int\frac{2axdx\cos^{-1}\left(\frac{a}{x}\right)}{\sqrt{(x^2-a^2)}};$$

$$\int\frac{2axdx\cos^{-1}\left(\frac{a}{x}\right)}{\sqrt{(x^2-a^2)}}=2a\sqrt{(x^2-a^2)}\cos^{-1}\left(\frac{a}{x}\right)-2a^2\log x;$$

$$\therefore\ \int 2xdx\left\{\cos^{-1}\left(\frac{a}{x}\right)\right\}^2=x^2\left\{\cos^{-1}\left(\frac{a}{x}\right)\right\}^2-2a\sqrt{(x^2-a^2)}\cos^{-1}\left(\frac{a}{x}\right)+2a^2\log x.$$

The other integrations are easy; performing them and substituting in (2), we have, finally,

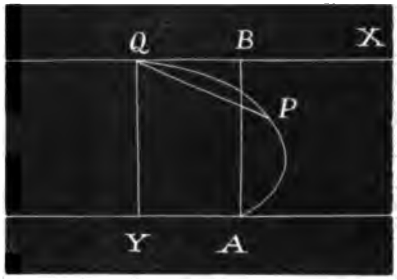
$$p=\frac{3a^2}{\pi^2(r^2-a^2)}-\left(\frac{2a^2}{\pi^2(r^2-a^2)}\right)\log\left(\frac{a}{r}\right)-\frac{2a(4r^2-a^2)\cos^{-1}\left(\frac{a}{r}\right)}{\pi^2(r^2-a^2)\sqrt{(r^2-a^2)}}+\frac{r^2(4r^2-a^2)\left\{\cos^{-1}\left(\frac{a}{r}\right)\right\}^2}{\pi^2(r^2-a^2)^2}.$$

Solved also by Professor *E. B. Seitz* and *Walter Siverly*.

133.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A duck swims across a river a rods wide, always aiming for a point in the bank b rods up stream from a point opposite the place she started from. The velocity of the current is v miles an hour, and the duck can swim n miles an hour in still water. Required the equation of the *curve* the duck describes in space, and the distance she swims in crossing the river.

I.—Solution by WALTER SIVERLY, Oil City, Venango Co., Pa.; D. J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Pa.; and E. A. BOWSER, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, New Jersey.



Let A be the point of starting, B the point on the other side of the river opposite A, Q the point toward which the duck swims, QX, QY the axes of x, y, QX being measured down the river and QY directly across it, P the position of the duck at the time t after starting, $AB = a$, $QB = b$, $\angle PQX = \theta$, x and y the co-ordinates of P.

Resolving velocities in the directions of the axes x, y,

$$\frac{dx}{dt} = v - n\cos\theta, \quad \frac{dy}{dt} = -n\sin\theta; \text{ whence } \frac{dy}{dx} = -\frac{n\sin\theta}{v - n\cos\theta}.$$

Putting $\frac{v}{n} = e$, $\frac{dy}{dx} = -\frac{\sin\theta}{e - \cos\theta} = -\frac{y}{e\sqrt{(x^2+y^2)} - x}$;

whence $\quad xdy - ydx = e\sqrt{(x^2+y^2)}dy \ldots\ldots\ldots\ldots (1).$

Let $x = yz$, then $dx = ydz + zdy$ and by substitution (1) becomes $\quad e\sqrt{(1+z^2)}dy = -ydz$, $\quad$ whence

$$\frac{edy}{y} = -\frac{dz}{\sqrt{(1+z^2)}}. \quad \text{Integrating, } e\log y = \log[(1+z^2) - z)] + C = \log\left(\frac{\sqrt{(x^2+y^2)} - x}{y}\right) + C.$$

When $x = b$, $y = a$; $\therefore \quad e\log\left(\frac{y}{a}\right) = \log\left(\frac{a[\sqrt{(x^2+y^2)} - x]}{y[\sqrt{(a^2+b^2)} - b]}\right)$;

whence $$[\sqrt{(a^2+b^2)} - b]\left(\frac{y}{a}\right)^{1+e} = \sqrt{(x^2+y^2)} - x \ldots\ldots\ldots\ldots (2).$$

Taking the reciprocals of (2) and then subtracting (2) from the result,

$$2x = [\sqrt{(a^2+b^2)} + b]\left(\frac{y}{a}\right)^{1-e} - [\sqrt{(a^2+b^2)} - b]\left(\frac{y}{a}\right)^{1+e} \ldots\ldots\ldots\ldots (3),$$

which is the equation to the curve.

Let s = the required distance; then

$$s = \int_0^a\left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}}dy = \tfrac{1}{2}\int_0^a\left[(1-e)^2\left(\frac{\sqrt{(a^2+b^2)}+b}{a^{1-e}}\right)^2 y^{-2e} + 2(1+e^2) + (1+e)^2\left(\frac{\sqrt{(a^2+b^2)}-b}{a^{1+e}}\right)^2 y^{2e}\right]^{\frac{1}{2}}dy.$$

Put $(1-e)^2\left(\frac{\sqrt{(a^2+b^2)}+b}{a^{1-e}}\right)^2 = p$, $2(1+e^2) = q$, $(1+e)^2\left(\frac{\sqrt{(a^2+b^2)}-b}{a^{1+e}}\right)^2 = r$, $y^{2e} = w$, $a^{2e} = w'$,

and $\frac{1-3e}{2e} = m$; then $s = \frac{1}{4e}\int_0^{w'} w^m\sqrt{(p + qw + rw^2)}dw$, which can be integrated by formulæ of reduction when $m = 0$ or any whole number, and when $\frac{n}{v}$ = any odd whole number greater than unity.

To find the time of crossing, we have $\frac{dy}{dt} = n\sin\theta = \frac{ny}{\sqrt{(x^2+y^2)}}$. Substituting the value of $\sqrt{(x^2+y^2)}$ obtained by adding (2) to its reciprocal,

$$\frac{dy}{y}\left\{[\sqrt{(a^2+b^2)} + b]y^{1-e}a^{e-1} + [\sqrt{(a^2+b^2)} - b]y^{1+e}a^{-1-e}\right\} = -2ndt.$$

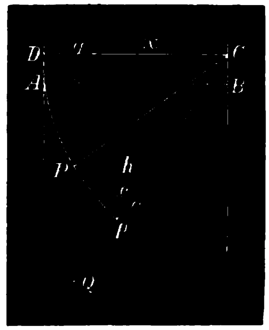
Integrating from $y = a$ to $y = 0$, putting t_1 = the time of crossing,

$$2nt_1 = \frac{[\sqrt{(a^2+b^2)} + b]a^2}{1-e} + \frac{[\sqrt{(a^2+b^2)} - b]a^2}{1+e}, \text{ whence } t_1 = \frac{\sqrt{(a^2+b^2)} + eb}{2n(1-e^2)} = \frac{n\sqrt{(a^2+b^2)} + bv}{n^2 - v^2}.$$

Cor.—If $b = 0$, the equation to the curve becomes $2x = a^e y^{1-e} - a^{-e}y^{1+e}$, and $t_1 = \frac{na}{n^2 - v^2}$.

II.—Solution by DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

Let P and p be consecutive points in the curve, origin at C, $Cg = x$, $gP = y$, $Ph = -dx$, $hp = dy$, $PQ = vdt$, $Qp = ndt$. From the two last we have $Qp = \frac{v}{n}PQ = mPQ$, putting $m = \frac{v}{n}$. Draw Pe perpendicular to CQ, and let c be the point where it intersects hp. The similar right-angled triangles CgP, Phc, cep and PeQ give $hc = -\frac{xdx}{y}$, $cp = dy - hc = \frac{ydx + xdy}{y}$, $ep = d\sqrt{(x^2+y^2)}$, $Pc = -\frac{\sqrt{(x^2+y^2)}dx}{y}$.



$$ce = \frac{xd\surd(x^2+y^2)}{y},\ \text{Pe} = \text{Pc} + ce = -\frac{\surd(x^2+y^2)dx}{y} + \frac{xd\surd(x^2+y^2)}{y},$$

$$\text{PQ} = \frac{\text{CP}}{\text{P}g} \times e\text{Q} = \frac{\surd(x^2+y^2)}{y}[d\surd(x^2+y^2) + m.\text{PQ}];$$

$$\therefore\ \text{PQ} = \frac{\surd(x^2+y^2)d\surd(x^2+y^2)}{y - m\surd(x^2+y^2)}.$$

Also, $\text{PQ} = \frac{\text{CP}}{\text{C}g} \times \text{Pe} = \frac{\surd(x^2+y^2)}{x}\left[\frac{xd(x^2+y^2) - \surd(x^2+y^2)dx}{y}\right].$

Making these values of PQ equal,

$$dy = \left(\frac{x^2+y^2}{y}\right)\frac{dx}{x} - \frac{m\surd(x^2+y^2)dx}{x} - \frac{xdx}{y}.$$

Transpose $\frac{xdx}{y}$, multiply by y, divide by $\surd(x^2+y^2)$, and we find

$$d\surd(x^2+y^2) = \frac{\surd(x^2+y^2)dx}{x} - \frac{mydx}{x},$$ which added to the preceding, observing that

$$\left(\frac{x^2+y^2}{y}\right)\frac{dx}{x} - \frac{xdx}{y} = \frac{ydx}{x},\ \text{gives}\ \frac{d[y+\surd(x^2+y^2)]}{y+\surd(x^2+y^2)} = (1-m)\frac{dx}{x}.$$

Integrating, observing that when $x = a$, $y = b$ and CA $= c$, we find

$$\left(\frac{x}{a}\right)^{1-m} = \frac{y+\surd(x^2+y^2)}{b+c},\ \text{which may be reduced to}\ 2y = (b+c)\left(\frac{x}{a}\right)^{1-m} - \frac{a^2}{b+c}\left(\frac{x}{a}\right)^{1+m}.$$

This problem was also very elegantly solved by Professor *E. B. Seitz.*

134.—Proposed by WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Auglaize County, Ohio.

Two equal rings begin to move freely from the extremity of the horizontal radius of a quadrant of a circle, one down the arc, the other down the chord; if the radius be 200 feet, how long after the motion begins will one ring be vertically above the other?

I.—Solution by CHARLES H. TUTTON, Wilkes Barre, Luzerne County, Pennsylvania; WALTER SIVERLY, Oil City, Venango County, Pennsylvania; and ABRAM R. BULLIS, Ithaca, Tompkins County, New York.

Take the origin at the extremity of the horizontal radius, from which the bodies start, x horizontal, y vertical and positive downwards.

The velocity at any point is by Mechanics $v = \surd(2gy)$, and we also have $ds = vdt$, s being the space described and t the time.

For arc, $y = \surd(2rx - x^2)$; $\therefore\ v = (2g)^{\frac{1}{2}}(2rx - x^2)^{\frac{1}{4}}$. $ds = \surd(dx^2 + dy^2) = \frac{rdx}{\surd(2rx - x^2)}$, whence for arc $dt = \frac{rdx}{(2g)^{\frac{1}{2}}(2rx - x^2)^{\frac{3}{4}}}$. Expand by Maclaurin's Theorem and integrate.

For arc, $t = \frac{r}{\surd(2g)}\left\{4\left(\frac{x}{8r^3}\right)^{\frac{1}{4}} + \frac{3}{5}\left(\frac{x^5}{128r^7}\right)^{\frac{1}{4}} + \frac{7}{24}\left(\frac{x^9}{16384r^{11}}\right)^{\frac{1}{4}} + \dots\right\} \dots(1)$. $\text{C} = 0$ because for $x = 0$, $t = 0$.

For chord, $y = x$, hence $v = \surd(2gx)$, $ds = \surd(dx^2 + dy^2) = \surd 2dx$, $dt = \frac{\surd 2dx}{\surd(2gx)}$ and $t = 2\sqrt{\left(\frac{x}{g}\right)} \dots(2)$.

By the question these times are equal, as also the values of x; hence equate (1) and (2) and divide by $x^{\frac{1}{4}}$;

$$1.11712 + 0.0004306x + 0.00000096x^2 + \dots = 0.3524x^{\frac{1}{4}},$$

whence $x^{\frac{1}{4}} = 3.3358$ nearly, $x = 123.822$ nearly, and from (2) $t = 3.922$ seconds nearly.

An elaborate solution was furnished by Professor *McAdam.*

135.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Three equal circles touch each other externally; find the average area of all the circles that can be drawn in the space enclosed by them.

I.—Solution by ABRAM R. BULLIS, Ithaca, Tompkins County, New York,

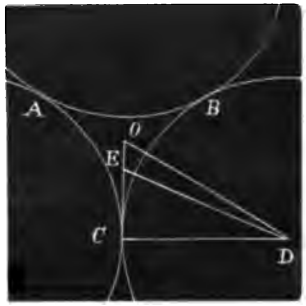
Let ABC be the space enclosed by the circles, D the center of one of them, CD $= a$, DE $= r$, $\angle$CDE $= \theta$ and $x =$ the radius of a circle having E for its center.

The limits of x are 0 and $r - a$; of r, a and $\frac{a}{\cos\theta} = r'$; of θ, 0 and $\frac{1}{6}\pi$, and the required average area is

$$\frac{\int_0^{\frac{1}{3}\pi}\int_a^{r'}\int_0^{r-a}\pi x^2 d\theta r dr dx}{\int_0^{\frac{1}{3}\pi}\int_a^{r'}\int_0^{r-a} d\theta r dr dx} = \frac{\frac{1}{3}\pi\int_0^{\frac{1}{3}\pi}\int_a^{r'}(r^4-3ar^3+3a^2r^2-a^3r)d\theta dr}{\int_0^{\frac{1}{3}\pi}\int_a^{r'}(r^2-ar)d\theta dr}$$

$$= \frac{\frac{1}{3}\pi a^2\int_0^{\frac{1}{3}\pi}\left(\frac{1}{5\cos^5\theta}-\frac{3}{4\cos^4\theta}+\frac{1}{\cos^3\theta}-\frac{1}{2\cos^2\theta}+\frac{1}{20}\right)d\theta}{\int_0^{\frac{1}{3}\pi}\left(\frac{1}{3\cos^3\theta}-\frac{1}{2\cos^2\theta}+\frac{1}{6}\right)d\theta}$$

$$= \frac{\pi a^2}{60}\left(\frac{6\pi+308-320\sqrt{3}+207\log 3}{\pi+4-6\sqrt{3}+3\log 3}\right).$$



Very elegant solutions received from Messrs. *Brown*, *Seitz* and *Siverly*. Mr. *Brown's* solution will be published in No. 5.

136.—Proposed by JOHN W. BERRY, Pittston, Luzerne County, Pennsylvania.

A smooth, straight, thin tube is balancing horizontally about its middle point, and a particle whose weight is one-nth that of the tube is shot into it horizontally with such a given velocity that it just arrives at the middle point of the tube. Find the angular velocity communicated to the tube.

I.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let $2a$ = the length of the tube, m its mass, k its radius of gyration, θ its inclination to the horizon at any time t, v the given velocity of projection of the particle, ω the required angular velocity, x and y its horizontal and vertical co-ordinates at the time t, the origin being at any point in the horizontal line passing through the initial position of the tube.

By the principle of *Vis Viva*,

$$\frac{m}{n}\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)+mk^2\frac{d\theta^2}{dt^2} = C-\frac{2m}{n}gy.$$

Initially $\frac{dx}{dt}=v,\quad \frac{dy}{dt}=0,\quad \frac{d\theta}{dt}=0,\quad y=0;\quad \therefore C=\frac{mv^2}{n},$ and

$$\frac{m}{n}\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}\right)+mk^2\frac{d\theta^2}{dt^2} = \frac{m}{n}(v^2-2gy).$$

When the particle reaches the middle of the tube,

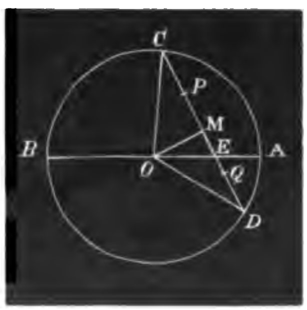
$$\frac{dx}{dt}=0,\ \frac{dy}{dt}=0,\ \frac{d\theta}{dt}=\omega,\ y=0;\ \text{also } k^2=\tfrac{1}{3}a^2,\ \therefore\ \tfrac{1}{3}ma^2\omega^2=\frac{mv^2}{n};\ \text{whence } \omega=\frac{v}{a}\sqrt{\left(\frac{3}{n}\right)}.$$

Solved also by Professor *De Volson Wood*.

137.—Proposed by E. B. SEITZ, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

Two points are taken at random in the surface of a circle, but on opposite sides of a given diameter; find (1) the chance that the chord drawn through them does not exceed a line of given length, and (2) the average length of the chord.

I.—Solution by the PROPOSER.



Let AB be the given diameter, O the center of the circle, P, Q the two random points, and CD the chord through them. Draw OM perpendicular to CD.

Let OA $=r$, EP $=x$, EQ $=y$, EC $=u$, ED $=u'$, $\angle$COM $=\theta$, $\angle$EOM $=\varphi$, and let 2β = the angle formed by the radii drawn to the extremities of a chord equal in length to the given line. Then we have CD $=2r\sin\theta$, $u=r(\sin\theta+\cos\theta\tan\varphi)$, $u'=r(\sin\theta-\cos\theta\tan\varphi)$; an element of the circle at P is $r\sin\theta d\theta dx$, and at Q it is $(x+y)d\varphi dy$.

1. The limits of θ are 0 and β; those of φ, $-\theta$ and θ, and doubled; of x, 0 and u; and of y, 0 and u'. Hence, since the whole number of ways the two points can be taken is $\frac{1}{4}\pi^2r^4$, the required chance is

$$p = \frac{8}{\pi^2r^4}\int_0^\beta\int_{-\theta}^{\theta}\int_0^u\int_0^{u'} r\sin\theta\, d\theta d\varphi dx(x+y)dy = \frac{4}{\pi^2r^3}\int_0^\beta\int_{-\theta}^{\theta}\int_0^u[(x+u')^2-x^2]\sin\theta d\theta d\varphi dx$$

$$= \frac{8}{\pi^2}\int_0^\beta\int_{-\theta}^{\theta}(1-\cos^2\theta\sec^2\varphi)\sin^2\theta d\theta d\varphi = \frac{16}{\pi^2}\int_0^\beta(\theta-\sin\theta\cos\theta)\sin^2\theta d\theta = \left(\frac{2\beta-\sin 2\beta}{\pi}\right)^2.$$

Cor. When the given line is equal to the radius of the circle, $2\beta = 60^\circ, = \frac{1}{3}\pi$, and $p = \left(\frac{1}{3} - \frac{3\sqrt{3}}{4\pi}\right)^2$.

2. The limits of θ are 0 and $\frac{1}{2}\pi$, and those of φ, x and y the same as above. Hence the required average is

$$\varDelta = \frac{8}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{\theta}\int_0^{u}\int_0^{u'} 2r^2\sin^2\theta d\theta d\varphi dx(x+y)dy = \frac{8}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{\theta}\int^{u}[(x+u')^2 - x^2]\sin^2\theta d\theta d\varphi dx$$

$$= \frac{16r}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_{-\theta}^{\theta}(1-\cos^2\theta\sec^2\varphi)\sin^3\theta d\theta d\varphi = \frac{32r}{\pi^2}\int_0^{\frac{1}{2}\pi}(\theta - \sin\theta\cos\theta)\sin^3\theta d\theta = \frac{832r}{45\pi^2}.$$

The first part of this problem is the same as Quest. 1849 in the *Educational Times*, to which erroneous solutions were given in vol. viii of the *Reprint*. pp. 92—4, by *Stephen Watson* and *Samuel Roberts*. Mr. *Roberts* has furnished a correct solution which will be published in No. 5. *Walter Siverly* also sent an elegant solution which we intend to publish.

☞ For want of time, and space, we very reluctantly omit the solutions of Problems 138—147; they will be published in No. 5.

List of Contributors to the Senior Department.

WALTER SIVERLY, Oil City, Venango County, Pennsylvania, solved all the problems but 131 and 142. E. B. SEITZ, Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo., solved 111, 113, 115, 120, 121, 122, 124, 127, 128, 130, 132, 133, 135, 137, 142, 143. 144 and 147. A. R. BULLIS, Ithaca, N. Y., solved 111, 113, 114, 115, 118, 121, 123, 128, 130, 131, 134, 135, 138 and 146. DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, N. J., solved 111, 113, 116, 117, 120, 122, 131, 133, 136, 139, 145 and 147. WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Auglaize Co., O., solved 111, 113, 114, 115, 121, 123, 128, 138 and 140. CHAS. H. TUTTON, Wilkes Barre, Luzerne Co., Pa., solved 111, 112, 119, 120, 121, 128, 134 and 133. CHAS. H. KUMMELL, Assistant Engineer, U. S. Lake Survey, Detroit, Mich., solved 111, 116, 121, 128, 133, 140 and 147. Dr. DAVID S. HART, M. A., Stonington, Conn., solved 112, 113, 114, 115, 123 and 129. W. E. HEAL, Wheeling, Ind., solved 111, 112, 115, 120, 121 and 146. D. J. MCADAM, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Pa., solved 113, 117, 122, 128, 133 and 134. REUBEN DAVIS, Bradford, Stark Co., Ill., solved 112, 114, 115, 123, 126 and 129. FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Pa., solved 113, 114, 115, 116 and 123. E. A. BOWSER, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, N. J., solved 113, 120, 121, 133 and 147. H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C., solved 130, 138, 146 and 147. LUCIUS BROWN, Hudson, Mass., solved 120, 121, 123 and 135. Prof. W. P. CASEY, C. E., San Francisco, Cal., solved 111, 116, 119 and 128. WALTER S. NICHOLS, Editor *Insurance Monitor*, New York, N. Y., solved 111, 113, 116 and 120. JULIAN A. POLLARD, Windsor, Vt., solved 111, 112, 119 and 120. J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa., solved 113, 115 and 118. GAVIN SHAW, Kemble, Ontario, Canada, solved 111, 116 and 126. W. L. HARVEY, Maxfield, Me., solved 111, 113 and 128. HENRY HEATON, Atlantic, Iowa, solved 138, 140 and 147. O. D. OATHOUT, Read, Iowa, solved 111, 116 and 133. J. J. SYLVESTER, LL. D., F. R. S., Corresponding, Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Md., solved 120 and 141. SAMUEL ROBERTS, M. A., F. R. S., Member of the London Mathematical Society, London, England, solved 127 and 137. F. P. MATZ, M. E., M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Ga., solved 121 and 138. Prof. E. J. EDMUNDS, B. S., Principal of Academic School No. 3, New Orleans, La., solved 111 and 121. Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Me., solved 112 and 114. SYLVESTER ROBINS, North Branch Depot, N. J., solved 112 and 115. E. P. NORTON, Allen, Mich., solved 111 and 113. MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C., solved 111 and 116. BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Mass., solved 145. Dr. JOEL E. HENDRICKS, M. A., Editor of the *Analyst*, Des Moines, Iowa, solved 117. WILLIAM T. MAGRUDER, Stevens Institute, Hoboken, N. J., solved 116. K. S. PUTNAM, Rome, N. Y., solved 113. Prof. DAVID TROWBRIDGE, M. A., Waterburg, N. Y., solved 122. ROBINS FLEMING, Readington, N. J., solved 112. Prof. ASHER B. EVANS, M. A., Principal Lockport Union School, Lockport, N. Y., solved 114. L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C., solved 111. G. W. HILL, Ph. D., Nautical Almanac Office, Washington, D. C., solved 117.

The first prize is awarded to WALTER SIVERLY, Oil City, Venango Co., Pa., and the second prize to CHARLES H. KUMMELL, Assistant Engineer, U. S. Lake Survey, Detroit, Mich.

PROBLEMS.

175.—Proposed by W. L. HARVEY, Maxfield, Penobscot County, Maine.

A man buys a farm for $4000 and agrees to pay it in 4 equal annual installments, interest at 5 per cent. per annum, compounded every instant. Required the annual payment.

176.—Proposed by Prof. E. J. EDMUNDS, B. S., Principal of Academic School No. 3, New Orleans, Louisiana.

The sides of an inscribed quadrilateral are the roots of a given equation of the fourth degree whose roots are all real and positive. Find the area of the quadrilateral, and the radius of the circumscribing circle, in functions of the coefficients of the equation.

177.—Proposed by JOSEPH H. KERSHNER, Professor of Mathematics, Mercersburg College, Mercersburg, Franklin Co., Pa.

Find three square numbers the ratio of whose sum and product shall be a square.

178.—Proposed by Prof. W. P. CASEY, C. E., San Francisco, California.

Given the three lines joining the remote angles of the equilateral triangles described on the sides of a triangle, to construct the triangle and find its sides.

179.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

In a series of rational right-angled triangles where every hypothenuse is unity, each leg of the nth triangle contains n decimal places. Find the legs of the first 25 triangles, and those of the 50th one.

180.—Proposed by Dr. JOHN BELL, Manchester, Hillsborough County, New Hampshire.

One-third of all the apples on a certain tree are rotten, and one-fourth of all the apples on the same tree are wormy. What are the respective chances that an apple taken at random from the tree will be (1) sound, (2) rotten, (3) wormy, (4) both rotten and wormy?

181.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

A polygon is both inscriptible and circumscriptible, radius of inscribed circle r, of circumscribed circle R and distance between the centers of these circles d; then if the polygon be a triangle,

$$\frac{1}{R+d}+\frac{1}{R-d}=\frac{1}{r};$$

if the polygon be a quadrilateral, $$\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2}=\frac{1}{r^2}.$$

What is the corresponding relation for a pentagon?

182.—Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., Edinburg, Scotland.

Prove that

$$\log[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{2}} = \log a+\frac{b}{a}\cos(\alpha-\beta)-\tfrac{1}{2}\frac{b^2}{a^2}\cos 2(\alpha-\beta)+\tfrac{1}{3}\frac{b^3}{a^3}\cos 3(\alpha-\beta)-\text{etc.}$$

183.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A clock which indicates correct time at the level of the ocean is carried to the top of a mountain near by, the ascent occupying h hours. On arriving at the top of the mountain the clock was found to be m minutes too slow. Required the hight of the mountain.

184.—Proposed by Dr. SAMUEL HART WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Given the latitude $=\lambda=42^\circ\ 30'$ N., the sun's declination $=\delta=20^\circ$ N., his radius $=r=16'$, and a vertical wall running S. 10° W., to find when the sun will first shine on the west side of the wall, the points from the sun's north point of ingress and egress on the wall, the altitudes of those points, and of the sun's center.

185.—Proposed by SAMUEL ROBERTS, M. A., F. R. S., Member of the London Mathematical Society, London, England.

Show that if the system of equations

$$ax^2+bxy+cy^2=u^2, \qquad ax^2-bxy-cy^2=v^2,$$

are resoluble by integer values of x, y, u, v, (zero excluded,) not having a common factor greater than unity, successive solutions can be obtained of the same character.

186.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

Inscribe in any plane triangle three circles each tangent to two sides and the two other circles.

187.—Proposed by Dr. JOEL E. HENDRICKS, M. A., Editor of the *Analyst*, Des Moines, Polk County, Iowa.

The base of a hemisphere, whose radius is ten feet, rests on a horizontal plane, and a point A on the surface of a sphere whose radius is one foot is in contact with the vertex of the hemisphere.

If, from a slight disturbance of the equilibrium, the sphere is caused to *roll* off the hemisphere in the direction of the plane of a great circle through the points A and B on the surface of the sphere, and if the point B strikes the horizontal plane at a point D, distant CD from the center of the hemisphere; it is required to find the length of the line CD, and the relative position of the points A and B on the surface of the sphere.

188.—Proposed by DANIEL KIRKWOOD, LL. D., Professor of Mathematics, Indiana State University, Bloomington, Ind.

If a mass of matter be projected vertically upward from Jupiter's equator with a velocity just sufficient to carry it to a hight of 450000 miles, find the path it will describe, or determine the circumstances of its motion; the equatorial diameter of Jupiter being 90000 miles, and its time of rotation 9 hours and 55 minutes.

189.—Proposed by Prof. HUGH S. BANKS, Instructor in English and Classical Literature, Newburg, Orange County, N. Y.

How many spheres one inch in diameter can at the same time touch the surface of a sphere one foot in diameter?

190.—Proposed by F. P. MATZ, M. E., M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Carroll County, Georgia.

A given rectangle is divided at random by a line parallel to one of its diagonals, and then two points are taken at random within the rectangle; find the chance that both points are on the same side of the dividing line.

191.—Proposed by REUBEN DAVIS, Bradford, Stark County, Illinois.

Find three square numbers such that if to each its root be added, and from each its root be subtracted, the three sums and three remainders shall all be rational squares.

192.—Proposed by DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

Three concentric shells of infinitesimal thickness and uniform density, radii r, are in contact but move independently of each other without friction with angular velocities ω, ω', ω''. Their axes of revolution are all in one plane but make angles α, β, γ with a fixed line.

Suddenly the three shells become one solid. Required the position of the new axis, and the angular velocity.

193.—Proposed by WILLIAM WOOLSEY JOHNSON, Professor of Mathematics, St. John's College, Annapolis, Maryland.

If we write $(x-iy)^n = A+Bi$, $(i=\sqrt{-1},)$ show that the expression $A+qB$, where q is any real numerical quantity, is the product of n real factors linear in x and y, and find these factors.

194.—Proposed by JOHN W. BERRY, Pittston, Luzerne County, Pennsylvania.

Two points are taken at random on the surface of the earth, but on opposite sides of the equator; find the average distance between them (1) supposing the earth a sphere, radius r, and (2) supposing it an oblate spheroid, semi-axes a and b.

195.—Proposed by GEORGE LILLEY, M. A., Corning, Adams County, Iowa.

A first integral of the differential equation

$$y-x\frac{dy}{dx}+\tfrac{1}{2}x^2\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}-x\frac{d^2y}{dx^2}\right)^2-\left(\frac{d^2y}{dx^2}\right)^2=0 \text{ is } y+(\tfrac{1}{2}a-a^2)x^2-(1-2a)x\frac{dy}{dx}-a^2-\left(\frac{dy}{dx}\right)^2=0.$$

Show that the complete primitive is $y=\frac{1}{2}ax^2+bx+a^2+b^2$, a and b being arbitrary constants.

196.—Proposed by W. A. KITE, M. A., Professor of Mathematics, Greeneville and Tusculum College, Tusculum, Tennessee.

Required the maximum ellipse that may be drawn within a trapezoid, parallel sides a and b and length of each of the oblique sides c.

197.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

To find three numbers such that if the sum of their cubes be either added to, or subtracted from, the square of each of them the sums and remainders shall be squares.

198.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

At the center of a square park, sides a, stands a square tower, sides b. Two persons are in the park; find the chance that they can see each other.

199.—Proposed by L. G. BARBOUR, Professor of Mathematics, Central University, Lexington, Fayette County, Kentucky.

A thousand foxes start from a point in a field and at the same instant a thousand hounds start in pursuit each of a fox, from another point situated a rods distant from the first point. The foxes run in straight lines in divergent directions, at the same uniform rates of speed, and the hounds at a speed m times as great. Required the equation to the *curve* upon which all the foxes will be caught, and the length of the curve.

200.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

From one corner of a square field a projectile is thrown at random with a given velocity which is such that the greatest range of the projectile is equal to the diagonal of the field; find the chance of its falling in the field.

201.—Proposed by E. B. SEITZ, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

A cube is thrown into the air and a random shot fired through it; find the chance that the shot passes through opposite faces.

202.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Massachusetts.

Find a curve which is similar to its own evolute.

203.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A cylinder, radius r, *rolls* down the surface of another one, radius R, on a horizontal plane, the surfaces of both cylinders and plane being rough enough to secure perfect rolling. Determine the motion of the cylinders, the line of separation, and the path of a given point in the axis of the upper cylinder.

204.—Proposed by E. B. Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

Find the average area of the triangle formed by joining three points taken at random in the surface of a given ellipse.

205.—Proposed by F. P. Matz, M. E., M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Carroll County, Georgia.

Find the average area of the triangle formed by joining three points taken at random (1) in the surface of a given quadrant, and (2) in the surface of a given semicircle.

206.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A point is taken at random in one of the longer sides of a rectangle and a line drawn from it at random to the opposite side, and then two points are taken at random in the surface of the rectangle; find the chance that both points are on the same side of the line.

207.—Proposed by Walter Siverly, Oil City, Venango County, Pennsylvania.

An ellipse, major axis $2a$ and eccentricity e, revolves about a tangent at the extremity of the major axis. The major axis increases uniformly from $2a$ to $2(a+n)$ while revolving through an angle β, the ellipse increasing in such a way that e remains constant. Required the volume of the solid generated.

208.—Proposed by Rev. W. J. Wright, M. A., Ph. D., Member of the London Mathematical Society, Jenkintown, Pa.

Show by actual construction that two plane surfaces lying in one plane can be attached in such a mannner as to be capable of being spread out into two sheets of a developable surface in which one edge of the former plane becomes an edge of regression and a curve of double curvature.

Show also how to obtain the equation of the developable surface.

209.—Proposed by E. B. Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

Two small circles are drawn on the surface of a sphere so as to intersect; find the average of the spheric surface common to the two segments cut from the sphere.

210.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Three circles, radii a, b, c, touch each other externally; find the average area of all the circles that can be drawn in the space enclosed by them.

211.—Proposed by E. A. Bowser, Professor of Mathematics and Engineering, Rutgers Scientific School, New Brunswick, Middlesex County, New Jersey.

A sphere, whose equation is $x^2+y^2+z^2=b^2$, has its center coinciding with that of an ellipsoid whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where $a>b>c$. Find (1) the surface and (2) the volume of the solid common to both by the formulas $S=\int\int dxdy\left(1+\frac{dz^2}{dx^2}+\frac{dz^2}{dy^2}\right)^{\frac{1}{2}}$. $V=\int\int\int dxdydz$.

212.—Proposed by B. B. Lakin, Streator, La Salle County, Illinois.

A pole 80 feet long was standing upright against a vertical wall. A monkey began to ascend the pole at a uniform velocity, and at the same instant the foot of the pole began to move out from the wall with the same uniform velocity, the monkey arriving at the other end at the moment that end reached the ground. Required the equation to the *curve* the monkey described in space, and the distance he moved.

213.—Proposed by Charles H. Kummell, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.

Find the surface of the solid whose volume is required in Problem 147.

214.—Proposed by D. J. McAdam, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Pa.

In Morin's apparatus for deducing experimentally the laws of falling bodies, the length of the revolving cylinder is a feet, its diameter $2b$ feet, and it revolves n times per minute. Required the equation of the *curve* which the pin that projects from the falling body traces on the cylinder, and its length.

215.—Proposed by Enoch Beery Seitz, Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.

Let A, B, C, D, E, F be six random points within a sphere; find the chance that the planes through A, B, C and D, E, F intersect within the sphere.

216.—Prize Problem. Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Pa,

Required (1) the equation to the *curve* described by a given point in the rod that connects the two driving wheels, and (2) the equation to the *curve* described by a given point in the rod that connects the forward driving wheel of a locomotive with the piston rod.

217.—Prize Problem. Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Pa.

Three equal coins are dropped horizontally at random, one at a time, into a circular box whose diameter is twice that of one of the coins. Find the chance that only one of the coins rests on the bottom of the box.

Solutions of these problems should be received by October 1, 1880.

EDITORIAL NOTES.

It is with feelings of deep regret we record the deaths of three of our former mathematical correspondents: Hon. JOSIAH SCOTT, Ex-Judge of the Supreme Court of Ohio, Bucyrus, Ohio; Hon. ROBERT M. DEFRANCE, Attorney at Law, Mercer, Pennsylvania; and JAMES CLARK, Wayne, Maine.

"JOSIAH SCOTT was born on the first day of December, 1803, in Washington County, Pennsylvania, on his father's farm, three miles from Cannonsburg, the seat of Jefferson College, where he was educated under the celebrated Dr. McMillan. He boarded at home, walked daily to and from college, and graduated in 1821 with the highest honors of his class.

He emigrated to Bucyrus, Crawford County, Ohio, when it was but a hamlet in the wilderness, and there he first engaged in the practice of the profession of which he afterwards became such a distinguished ornament. He was elected Judge of the Supreme Court of Ohio, and twice re-elected, remaining on the bench for fifteen successive years. In 1876 he was appointed Chief Justice of the Supreme Court Commission."

The above is condensed from a biographical sketch published in the *Yates County Chronicle* of July 31, 1879, in which it is stated that he died on the 15th, of hemorrhage of the bladder.

Judge SCOTT was a mathematician of marked ability. He was particularly skillful in the solution of Algebraic and Geometrical problems, especially those pertaining to the Diophantine Analysis, as may be seen by referring to his elegant solutions in the *Schoolday Visitor Magazine* and the *Yates County Chronicle.*

Hon. ROBERT M. DEFRANCE died on the 19th of August, 1879, at the age of 58 years and 28 days. In October, 1860, he was struck with paralysis and rendered unable to get about without the aid of crutches, but retained his mental faculties and attended to the duties of his office until in the winter or spring of 1878. Mr. DEFRANCE was a lawyer by profession, and was elected a member of the Constitutional Convention of Pennsylvania in 1872.

We are not at present in possession of information for a fuller biographical notice.

In mathematics we believe Mr. DEFRANCE was chiefly self-taught. He was a valued contributor to the mathematical department of the *Schoolday Visitor Magazine*, and to the *Schoolday Visitor Annual*. His solutions were always ingenious, and usually original in method.

"JAMES CLARK was born in 1794, at Ayr in Scotland, a place famous in song, and a familiar name to readers of Robert Burns. He did not in early life enjoy the advantages of education which are, or may be enjoyed by every one in our highly-favored country. He learned the trade of a muslin weaver at which he worked till the age of 14, when he was apprenticed on board a ship. In his 17th year his captain sent him to school to study navigation. The term occupied about five weeks, and this closed his school career.

He left his native country for the last time in 1818, and learned the trade of a cabinet maker with his brother's employer in Portland, Me. He also learned pattern making, which introduced him to the machinist business, and for six or seven years he was foreman of a large establishment in the city of Alton, Illinois."—[From the *Yates County Chronicle.*

Mr. CLARK was a self-made mathematician of more than ordinary ability. He was for many years a regular contributor to the *Maine Farmers' Almanac* over the signature of *Mechanic*. He was also a contributor to the mathematical departments of the *Schoolday Visitor Magazine* and the *Yates County Chronicle*, in the pages of which periodicals many of his fine solutions are to be found. He delighted in working out the solutions of curious and intricate problems in the Diophantine Analysis.

We are not informed of the date of the death of Mr. CLARK, but think it occurred sometime in 1879.

E. B. SEITZ of Greenville, O., has been elected Professor of Mathematics in the North Missouri State Normal School, Kirksville, Mo., and F. P. MATZ, M. E., M. S., of Reading, Pa., has been elected Professor of Pure and Applied Mathematics in Bowdon (State) College, Bowdon, Georgia, where they are now hard at work.

Our talented lady contributor, Miss CHRISTINE LADD, has been deservedly honored by the Johns Hopkins University. The trustees have voted her an honorary stipend and an invitation to continue her mathematical studies in that institution.

Miss LADD is a graduate of Vassar College, and is probably the best mathematician of the gentler sex in this country.

We take pleasure in acknowledging our indebtedness to Prof. E. B. SEITZ, Prof. DEVOLSON WOOD and WALTER SIVERLY for valuable assistance.

The following persons, and many others, have our thanks for the subscriptions procured by them: JOHN S. ROYER, Prof. E. B. SEITZ, Prof. W. P. CASEY, Prof. E. A. BOWSER, Prof. DEVOLSON WOOD, SYLVESTER ROBINS, REUBEN KNECHT, W. D. MCSWANE, A. R. BULLIS, and Prof. FRANK T. FREELAND of the University of Pennsylvania.

A few such "live" mathematicians in *every* state would soon secure for the VISITOR a *paying* circulation.

This No. of the VISITOR has been delayed some months in consequence of the sickness of the Editor, who has done all the type-setting with his own hands. He is not a practical printer, and never had set up a stickful of type till last May or June.

The cover, the first six pages and page 96 were printed by the Editor on a No. 3 Self-Inker Model Press; pages 93, 94 and 95 were printed at the Erie Morning Dispatch Office, and the remaining pages at the Erie Gazette Office.

We have decided to issue the VISITOR semi-annually, at $1.00 a year in advance. Single numbers will be supplied at 50 cents each. Back numbers can be obtained at the same price.

We hope to have No. 5 ready in July; it will contain the Solutions omitted in this No., some Additional Solutions, and a few Papers.

Contributors will please send Problems, Solutions and Articles intended for publication in No. 5 at as early a date as practicable.

Send subscriptions soon to ARTEMAS MARTIN, *Lock Box 11, Erie, Pa.*

NOTICES OF BOOKS AND PERIODICALS.

An Elementary Treatise on the Differential Calculus, Founded on the Method of Rates or Fluxions. By John Minot Rice, Professor of Mathematics in the United States Navy, and William Woolsey Johnson, Professor of Mathematics in Saint John's College, Annapolis, Maryland. Revised Edition. 8vo., pp. 469. New York: John Wiley & Son.

This is the most extensive work on the Differential Calculus yet published in this country. It is well supplied with interesting examples, and contains many excellent solutions.

Sections IV and V of Chapter II are worthy the special attention of the student.

A large amount of space is devoted to the geometrical applications of the Differential Calculus, including Curve Tracing and the discussion of Higher Plane Curves.

The typography is very fine, and we heartily commend the book to all who want a good text-book on the subject it so fully treats. It is to be hoped that the authors will soon follow it with an equally exhaustive treatise on the Integral Calculus.

The Elements of Co-ordinate Geometry, in Three Parts. I. Cartesian Geometry. II. Quaternions. III. Modern Geometry, and an Appendix. By DeVolson Wood, Professor of Mathematics and Mechanics in Stevens Institute of Technology. 8vo., pp. 329. New York: John Wiley & Son.

In Part I, which contains 228 pp., the subjects usually treated in works on Analytic Geometry are very clearly presented, and illustrated with a large number of well-chosen examples and solutions, some of which have been selected from the *Educational Times Reprint*, the *Wittenberger* and the VISITOR, and properly credited.

The Second Part, which contains 72 pp., is devoted to Quaternions, and the subject is treated in the most elementary manner. The examples are of the simplest kind, and intended "to explain and illustrate the principles and the character of the operations without taxing the ingenuity of the student in the mere solution of problems."

The Third Part is very brief—too brief, containing only 16 pp. It is devoted to Tangential and Trilinear Co-ordinates, and Abridged Notation, &c.

The mechanical execution is first class. We heartily commend the work.

Elements of the Differential Calculus, with Examples and Applications. A Text-Book by W. E. Byerly, Ph. D., Assistant Professor of Mathematics, Harvard University. 8vo., pp. 258. Boston: Ginn & Heath.

A good work, based on the Doctrine of Limits, well printed on heavy paper and supplied with a goodly number of examples.

The notation is somewhat peculiar. The differential coefficient $\frac{dy}{dx}$ is represented by D_xy. Although not in common use it has the advantage of being easier to print.

Principles of the Algebra of Logic, with Examples. By Alexander Macfarlane, M. A., D. Sc., F. R. S. E. Read before the Royal Society of Edinburg 16th December 1878 and 20th January 1879. 12mo., pp. 155. Edinburg, Scotland: David Douglas.

"It is the object of this little work to investigate the foundations of the analytical method of reasoning about Quality, with special reference to the principles laid down by Boole as the basis of his calculus, and to the observations which have been published by various philosophers concerning these principles." P. 2, Art. 5.

It will well repay a careful reading.

Principles of Physics, or Natural Philosophy; Designed for the Use of Colleges and Schools. By Benjamin Silliman, Jr., M. A., M. D., Professor of General and Applied Chemistry in Yale College. Second Edition, Revised and Rewritten. 8vo., pp. 710. 722 illustrations. New York and Chicago: Ivison, Blakeman, Taylor, & Company.

An extensive treatise, beautifully illustrated and well supplied with appropriate examples and problems.

The Collegiate Algebra; Adapted to Colleges and Universities. By James B. Thomson, LL. D., Author of New Mathematical Series, and Elihu T. Quimby, M. A., Late Professor of Mathematics in Dartmouth College. 12mo., pp. 308. New York and Chicago: Clark & Maynard.

The definitions and rules are clear and concise; the demonstrations simple, rigorous and logical. The examples are numerous and good, but there is a lack of illustrative solutions. Besides the subjects usually found in works on Algebra, there are chapters on Infinitesimal Analysis and Loci of Equations, and two pages on Probabilities.

The Notes at the close of the book contain brief biographical sketches of several world-famous mathematicians.

Tracts Relating to the Modern Higher Mathematics. Tract No. III. Invariants. By W. J. Wright, Ph. D., Member of the London Mathematical Society. 8vo., pp. 75. London: C. F. Hodgson & Son.

A valuable addition to the literature of the subject. In the limited space of 75 pages Dr. Wright has treated in a very clear and forcible manner the important subject of Invariants.

The Modern Higher Algebra is beginning to receive some attention in this country, and the thanks of American students are due the gifted author for placing within their reach, at so small a cost, the results of the researches of the Great Masters.

Thermodynamics. By Henry T. Eddy, C. E., Ph. D., University of Cincinnati. 18mo., pp. 182. Reprinted from Van Nostrand's Magazine. New York: D. Van Nostrand.

A New, Simple, and Complete Demonstration of the Binomial Theorem and Logarithmic Series. By J. W. Nicholson, M. A., Author of the Formula of Right-Angled Triangular and Circular Functions, and Professor of Mathematics, Louisiana State University. Baton Rouge: Louisiana Capitolian Print.

The Educational Times, and Journal of the College of Preceptors. London: C. F. Hodgson & Son.

The valuable Mathematical Department of three or four double-column quarto pages is ably edited by W. J. C. Miller, B. A., Registrar of the General Medical Council. Among its contributors are numbered many of the leading mathematicians of the world. About 25 problems are proposed in each No., and about half as many solutions are given.

Mathematical Questions, with their Solutions, Reprinted from the Educational Times. Same publishers. Issued in half-yearly volumes of 112 pp., 8vo., boards.

The *Reprint* contains, besides the mathematics published in the *Times*, a large number of additional solutions and papers. Vol. XXX contains 12 papers and solutions of 84 problems; Vol. XXXI contains 6 papers and solutions of 98 problems. Some of the papers relate to important and disputed points in the Theory of Probability. Vol. XXXII is announced as ready.

The Editor of the VISITOR can furnish the *Times* at \$2.00 a year, and the *Reprint* at \$1.75 per volume.

The Analyst. A Journal of Pure and Applied Mathematics. Edited and published by J. E. Hendricks, M. A., Des Moines, Iowa. Bi-monthly. $2.00 a year.

Nos. 1 and 2 of Vol. VII contain valuable articles and elegant solutions. The *Analyst* is ably conducted and should receive the cordial support of every mathematician.

The Indiana Mathematician. Edited and published by W. DeKalb McSwane, Superintendent of Schools, Petersburg, Indiana. Published semi-annually at $1.00 a year. Single numbers 50 cents. No. 1, Vol. I, August, 1879. 4to., pp. 8.

A local mathematical periodical intended for circulation among the students and teachers of Indiana. Only the first No. has been received; it is well gotten up and neatly printed, and contains 87 problems for solution, some solutions and an article on the Doctrine of Chances. No. 2 is not out yet, (delayed by sickness,) but is expected in a month or two. We wish it success.

The School Visitor, Devoted to the Study of Mathematics and English Grammar. Edited by John S. Royer and Thomas Ewbank. Ansonia, O.: Published by John S. Royer at 60 cents a year.

The *School Visitor* is a new periodical, and very tastefully gotten up; Nos. 1 and 2 contain a number of interesting problems and solutions of an elementary character. Its cheapness and merit should ensure it a large circulation.

The Wittenberger. Springfield, O.: Wittenberg College. 10 Nos. in a year. $1.00 a year.

The excellent Mathematical Department, to which some of the best mathematicians contribute, is edited by William Hoover, Superintendent of Schools, Wapakoneta, O.

Barnes' Educational Monthly. New York: A. S. Barnes & Co. $1.50 a year.

A very valuable educational magazine, and should be read by all teachers. It has an interesting Mathematical Department edited by Prof. F. P. Matz, now of Bowdon College, Georgia.

Educational Notes and Queries. Edited and published by Prof. W. D. Henkle, Salem, O. $100 a year.

Issued monthly except in the vacation months of July and August, and contains mathematical notes, problems and solutions, besides much valuable general information.

New-England Journal of Education. Weekly. Boston and Chicago: Published under the Auspices of the American Institute of Instruction and the Teachers' Associations of the New-England States. $3.00 a year; $2.50 if paid in advance.

One of the best educational journals in the country, and should be liberally patronized.
The Mathematical Department is edited by Prof. E. T. Quimby, but the publisher has "squeezed" the life almost out of it.

The Yates County Chronicle. Weekly. Penn Yan, N. Y.: Chronicle Publishing Co. $2.00 a year.

The interesting and valuable Mathematical Department continues under the able and efficient management of Dr. Wright.

The Canada School Journal. Monthly. Toronto, Ontario, Canada. $1.00 a year in advance.

A large 2?-page educational monthly crammed full of good matter. It has an able Mathematical Department edited by Alfred Baker, M. A.

The Pennsylvania School Journal. Monthly. J. P. Wickersham, Editor. Lancaster: J. P. Wickersham & Co. $1.60 a year.

One of the best of the educational monthlies. It has no department of mathematics, but scientific articles and examination problems are occasionally published.

American Journal of Education. Monthly. Saint Louis. J. B. Merwin. $1.60 a year.

A "live" periodical deserving a wide circulation among the teaching fraternity.

The Maine Farmers' Almanac. 1880. Hallowell, Me.: Masters & Livermore. Price 10 cents.

This excellent little annual contains the answers and solutions to the Puzzles, Riddles, Mathematical Questions, &c., propounded last year, and a new lot for solution in the next number.

CORRIGENDA.

No. 2.

Page 28, lines 22, 25 and 26, *for* "E(c)" *read* $\frac{m}{n}$E(c).

No. 3.

Page 55, solution of Problem 50, line 6, numerator of the fraction, *for* "nPc" *read* nP_c.

Page 77, second line of first solution, *for* "D" *read* B.

No. 4.

Page 90, solution of Problem 95, numerator of the value of $\cos\phi'$, *for* "x^2" *read* r^2.

Page 94, solution of Problem 107, line 8 from the bottom, *for* "2.5(1.2² - .5²)" *read* 25(1.2² - .5²).

Page 98, Problem 164, *for* "81 yards" *read* 81 rods; Problem 166, omit all after "b - 8° 48′ 30″" and insert "E".

Page 102, solution of Problem 117, lines 4 and 7, *for* "π" *read* π^2; and in line 4 from bottom *read* 0.685766 instead of the result given there.

Page 104, Problem 121, numerator of exponent of (2π), *for* "$1-n$" *read* $n-1$; last line of solution, *for* "vol. i" *read* vol. ii.

Page 108, solution of Problem 127, next to the last line, *for* "$\cos\phi$" *read* $\cos\theta$.

Page 109, solution of Problem 129, line 3, *for* "$s - s^3$" *read* $s - 3s^3$.

Page 110, solution of Problem 115, line 3, numerator of 14th convergent, *for* "9" *read* 4.

Page 114, line 1, *for* "$p - \left(\frac{1}{3} - \frac{3\sqrt{3}}{4\pi}\right)^2$" *read* $p - \left(\frac{1}{3} - \frac{\sqrt{3}}{2\pi}\right)^2$.

Vol. I. JULY, 1880. No. 5.

THE MATHEMATICAL VISITOR.

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ARTEMAS MARTIN, M. A.,

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Vol. I. *JULY, 1880.* *No. 5.*

JUNIOR DEPARTMENT.

Solutions of Problems Proposed in Nos. 1, 2 & 3.

1.—Proposed by Prof. EDWARD BROOKS, M. A., Ph. D., Principal of Pennsylvania State Normal School, Millersville, Pa.

Two trains, one a and the other b feet long, move with uniform velocities on parallel rails; when they move in opposite directions they pass each other in m seconds, but when they move in the same direction the faster train passes the other in n seconds. Find the rate at which each train moves.

III.—Solution by K. S. PUTNAM, Rome, Oneida County, N. Y.; W. L. HARVEY, Maxfield, Maine; and WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Ohio.

Let $x=$ number of feet the faster train moved in one second, and $y=$ number of feet the other moved in the same time.

$m(x+y)=$ distance traveled by both trains while passing when moving in opposite directions;

$$\therefore\quad m(x+y)=a+b \quad\ldots\ldots\ldots (1).$$

The faster train will gain $x-y$ feet each second, and in n seconds it gains $n(x-y)$ feet;

$$\therefore\quad n(x-y)=a+b \quad\ldots\ldots\ldots (2).$$

From equations (1) and (2) we have

$$x+y=\frac{a+b}{m} \text{ and } x-y=\frac{a+b}{n};$$

whence

$$x=\tfrac{1}{2}\left(\frac{a+b}{m}+\frac{a+b}{n}\right), \qquad y=\tfrac{1}{2}\left(\frac{a+b}{m}-\frac{a+b}{n}\right).$$

9.—Proposed by Miss CHRISTINE LADD, B. A., Baltimore, Maryland.

Show that in any plane triangle

$$\frac{a\sin A+b\sin B+c\sin C}{a\cos A+b\cos B+c\cos C}=R\left(\frac{a^2+b^2+c^2}{abc}\right).$$

II.—Solution by the PROPOSER.

Since $R=\dfrac{a}{2\sin A}=\dfrac{b}{2\sin B}=\dfrac{c}{2\sin C}$, $a^2+b^2+c^2=2R(a\sin A+b\sin B+c\sin C)$.

As shown by J. J. Walker in the *Educational Times*, March, 1875,

$$r(a+b+c)=R(a\cos A+b\cos B+c\cos C).$$

Hence $\dfrac{a\sin A+b\sin B+c\sin C}{a\cos A+b\cos B+c\cos C}=\dfrac{a^2+b^2+c^2}{4rs},=R\left(\dfrac{a^2+b^2+c^2}{abc}\right)$ since $abc=4Rrs$.

51.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

There is a series of right-angled triangles whose legs differ by unity only. Calling the one whose sides are 3, 4, 5 the *first* triangle, it is required to find general expressions for the sides of the nth triangle, and compute the sides of the 100th triangle.

II.—Solution by K. S. PUTNAM, Rome, Oneida County, New York.

Let x = shorter leg, $x+1$ = longer leg, s = sum of legs and y = hypothenuse.

Then $x^2+(x+1)^2=y^2$, from which we obtain (1) $x=\frac{1}{2}[\sqrt{(2y^2-1)}-1]$, (2) $y=\sqrt{[\frac{1}{2}(s^2+1)]}$.

Now by trial the first four values of x that will satisfy the above equations are 0, 3, 20 and 119.

Hence we have	x	$x+1$	s	y
Primary or vanishing triangle	0	1	1	1
First triangle	3	4	7	5
Second triangle	20	21	41	29
Third triangle	119	120	239	169

An inspection of the above gives the following formulæ:

$$(3)\quad (x+1)_n=4y_{n-1}+(x+1)_{n-2},\qquad (4)\quad y_n=4s_{n-1}+y_{n-2}.$$

From (3) and (4) we obtain values for x, $x+1$, s and y for the 4th triangle, and as many more as are desired. We shall then find by inspection (5) $s_n=6s_{n-1}-s_{n-2}$.

Representing the sum of legs of the primary triangle by s, and the sums of the legs of the 1st, 2nd, 3d, etc., triangles by s_1, s_2, s_3, etc., we have by (5),

$$s_2=6s_1-s,\qquad s_3=35s_1-6s,\qquad s_4=204s_1-35s,\qquad s_5=1189s_1-204s.$$

And thus proceeding we may find the sum of legs and thence the sides of each triangle in the series in succession.

When a single triangle is required the process may be shortened. In solving for the 100th triangle by proceeding as above we find

$$(6)\ s_{20}=361786555939836s_1-620727596307771s,\qquad (7)\ s_{21}=2108646576008245s_1-361786555939336s.$$

Now $s=1$ and $s_1=7$ and we have $s_{20}=2470433131948081$, $s_{21}=14389730476117879$.

Substituting these values of s_{20} and s_{21} in place of s and s_1 respectively in (6) and (7) we have

$$s_{40}=5055923762956339922096065927393,\qquad s_{41}=29468083200663558275864384257639.$$

Substituting these values for s and s_1 in (6) and (7) we find s_{60} and s_{61} which substituted in (6) and (7) give s_{80} and s_{81}, and the values of s_{80} and s_{81} substituted in (6) give us

$$s_{100}=43339386297227576661095959458572614328030405537398930692163383984677691935393;$$

whence $x_{100}=21669693148613788330547979729286307164015202768699465346081691992338845992696$,

$(x+1)_{100}=21669693148613783330547979729286307164015202768699465346081691992338845092697$.

From (2) we have $y_{100}=\sqrt{\{\frac{1}{2}[(s_{100})^2+1]\}}$. Performing the operations indicated we get

$$y_{100}=30645573943232956180057972969833245887630954508753693529117371074705767728665.$$

Prof. *Edgar Haas*, *M. A.*, County Superintendent, Burlington County, Bordentown, N. J., and Mr. *Reuben Davis*, Bradford, Illinois, furnished elaborate solutions of this problem.

94.—Proposed by HENRY NICHOLS, Hampton, Rock Island County, Illinois; and Mrs. ANNA T. SNYDER, Chicago, Illinois.

It is required to divide a tapering board into two equal parts by sawing it across, parallel to the ends. Find the width at each end so that the lengths of the pieces will be expressed by rational numbers.

III.—Solution by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

Let ABCD represent the tapering board, EF the division line, AI = BK = x = length of the board, CI = DK = ax, EG = HF = bx, and AB = GH = IK = y.

From the similar triangles AIC, AGE we find AG = $\frac{bx}{a}$. Then, by Mensuration, $x(ax+y)$ = area of ABCD and $\frac{bx(y+bx)}{a}$ = area of ABEF;

$$\therefore\quad x(ax+y)=\frac{bx(y+bx)}{a};\quad \text{whence}\quad y=\frac{(a^2-2b^2)x}{2b-a},$$

where x may be any number, and a and b such that $a>2b^2$ and $2b>a$.

Let $a=\frac{1}{12}$, $b=\frac{1}{18}$, and we have $y=\frac{1}{36}x$. If now x be taken = 36 then will $y=1$ = AB, $y+2ax$ = 7 = CD and $y+2bx$ = 5 = EF.

97.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Solve, by quadratics, the equation $x^4-2ax^3-2abx+b^2=0$.

III.—Solution by LUCIUS BROWN, Hudson, Middlesex Co., Mass.; DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, N. J.; J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and CHAS. H. TUTTON, Wilkes Barre, Luzerne County, Pennsylvania.

The equation may be written $(x^4+b^2)-2ax(x^2+b)=0$; adding $2bx^2$ to each side, and then extracting the square root,

$$x^2+b=ax\pm x\sqrt{(a^2+2b)}, \qquad \text{or} \qquad x^2-[a\pm\sqrt{(a^2+2b)}]x=-b;$$

whence $$x=\tfrac{1}{2}\{a\pm\sqrt{(a^2+2b)}\pm\sqrt{[2a^2-2b\pm2a\sqrt{(a^2+2b)}]}\}.$$

IV.—Solution by F. P. MATZ, M. E., M. S., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Carroll Co., Georgia; WILLIAM HOOVER, Superintendent of Schools, and Mathematical Editor of the *Wittenberger*, Wapakoneta, Ohio; ABRAM R. BULLIS, Ithaca, N. Y.; W. E. HEAL, Wheeling, Ind.; Prof. FRANK ALBERT, Millersville, Lancaster Co., Pa.; and Prof. H. T. J. LUDWIG, Mount Pleasant, North Carolina.

By adding $\dfrac{2b}{x^2}$ to each member, the given equation can be put in the following form:

$$\left(1+\frac{b}{x^2}\right)^2-\frac{2a}{x}\left(1+\frac{b}{x^2}\right)=\frac{2b}{x^2}; \quad \text{whence} \quad 1+\frac{b}{x^2}=\frac{a\pm\sqrt{(a^2+2b)}}{x},$$

and $$x=\tfrac{1}{2}\{a\pm\sqrt{(a^2+2b)}\pm\sqrt{[2a^2-2b\pm2a\sqrt{(a^2+2b)}]}\}.$$

99.—Proposed by W. WOOLSEY JOHNSON, Member of the London Mathematical Society, Professor of Mathematics, St. John's College, Annapolis, Maryland.

The extremities of a line of fixed length which slides along a fixed line are joined to two fixed points. Find the locus of the intersection of the joining lines.

II.—Solution by the PROPOSER.

Taking the line joining the fixed points as axis of x and that on which the line of fixed length slides as axis of y, and denoting the distances of the fixed points from the origin by a and b, the fixed length by c and the distance of the extremity joined to $(a, 0)$ by the variable z, the equations of the straight lines

are $$\frac{x}{a}+\frac{y}{z}=1 \qquad \text{and} \qquad \frac{x}{b}+\frac{y}{z+c}=1.$$

Eliminating z we obtain

$$cx^2+(a-b)xy-c(a+b)x+abc=0,$$

an hyperbola whose asymptotes are

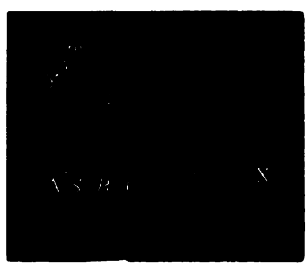
$$x=0 \quad \text{and} \quad cx+(a-b)y=c(a+b).$$

The curve passes through the given points and has the given line for an asymptote.

103.—Proposed by WILLIAM WOOLSEY JOHNSON, Member of the London Mathematical Society, Professor of Mathematics, Saint John's College, Annapolis, Maryland.

BR is an ordinate, from any point B of a circle, to the diameter passing through the fixed point A; and T is the intersection of the tangent at A with the radius produced through B. Find the locus of the intersection of AB and TR.

II.—Solution by LUCIUS BROWN, Hudson, Middlesex County, Massachusetts.



Let C be the center of the circle, and P the point whose locus is required. Draw PS perpendicular to AC, and let $AS=x$, $PS=y$, and $\angle ACB=\varphi$.

Then $x:y::r(1-\cos\varphi):r\sin\varphi::\sin\varphi:1+\cos\varphi$ .... (1)

and $x:r\tan\varphi-y::r(1-\cos\varphi):r\tan\varphi$ .... (2).

Therefore $r\tan\varphi-y:y::\tan\varphi:\sin\varphi::1:\cos\varphi$,

whence $r\sin\varphi-y\cos\varphi=y$, or $\sin\varphi:1+\cos\varphi::y:r$ ...... (3).

Comparing (3) with (1) gives $x:y::y:r$, or $y^2=rx$. Hence the locus is a parabola.

The methods employed by the *Proposer* and Professor *E. B. Seitz* are similar to this.

107.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

At a Firemen's Fair a silver trumpet is offered to the company exhibiting the ladder that can be used in the greatest number of streets and alleys for the purpose of reaching windows on either side without changing the location of its foot. All bases and perpendiculars must be rational lengths, and a company may include in their count dimensions having as many decimal places as their ladder has, but no more.

The "Hudsons" bring a ladder 65 feet long, the "Keystones" offer one $32\frac{1}{2}$ feet in length, and the "Delawares" show one of $42\frac{1}{4}$ feet. To whom must the trumpet be awarded, and on what count?

II.—Solution by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

Let x, y be the legs, and z the hypothenuse of a right-angled triangle; then $x^2+y^2=z^2$. The general values of x and y are $x=\frac{(p^2-q^2)z}{p^2+q^2}$, $y=\frac{2pqz}{p^2+q^2}$, the sum of whose squares is $=z^2$.

The formula $4t+1$ contains all prime numbers each of which is the sum of two squares. The product of any two such prime numbers is the sum of two squares in two ways; the product of any three is the sum of two squares in four ways; the product of any four is the sum of two squares in eight ways; and, generally, the product of any n such prime numbers is the sum of two squares in 2^{n-1} ways.

Let $z=65=5\times13$; then we have, by substituting p and q for the roots of the squares in 5 and 13, and of the two squares in 65, four sets of legs to the hypothenuse 65, or four triangles for the ladder of the "Hudsons".

Let $z=325=5\times5\times13$; then, substituting p, q for the roots of the squares in 5, 5, 25 and of the two squares each in 65 and 325, we have seven sets of legs to the hypothenuse 325, or, dividing by 10, we have 7 triangles whose legs correspond to the hypothenuse 32.5, the ladder brought by the "Keystones".

In a similar manner when $z=4225$ we find 12 triangles for the ladder exhibited dy the "Delawares".

Each of these triangles can be placed in three streets, having widths $2x$, $2y$ and $x+y$. Besides, placing each ladder in a perpendicular position close to the walls of the street, and moving the top so as to reach windows on the opposite side of the street, we shall have two streets with widths x, y. Then, for each triangle we shall have streets of the widths x, y, $2x$, $2y$ and $x+y$. Calling the number of triangles n, we have $5n=$ number of streets for the triangles taken separately. Combining the first triangle with each of the $n-1$ remaining; the second with the $n-2$ remaining, and so on; and, finally, combining the $(n-1)$th with th nth, we shall have $1+2+3+4+\ldots.+(n-1)$ for all the combinations. But, putting x, y, x', $y'=$ the legs of any two triangles in combination, we have $x+x'$, $y+y'$, $x+y'$, $y+x'$ for the widths of four streets; therefore we have $4[1+2+3+4+\ldots.+(n-1)]=$ number of streets for the triangles taken in combination, and $5n+4[1+2+3+4+\ldots.+(n-1)]=3n+2n^2=$ whole number of streets for n triangles.

Let $n=4$; then $20+24=44$ streets in which the "Hudsons" can place their ladder.

Let $n=7$; then $35+84=119$ " " "Keystones" " "

Let $n=12$; then $60+264=324$ " " "Delawares" " "

The trumpet must be awarded to the "Delawares".

NOTE ON THE SOLUTION OF 110.

By WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

The method of solution by Mr. *Putnam* is ingenious, and probably as good as any of the methods of approximation; but to suppose that any of those methods are "expeditious" for the purpose of extracting the square root is a mistake. For instance, to extract the square root of 2 to about 200 places, by the common method, the divisor increases from one figure to 100 figures, then decreases to one figure. If the figures of the root were all integers, this would give 100 figures in the divisor, (which need be written but once,) 200 figures in the root, and about 20000 figures in the rest of the work; in all, about 20300 figures.

By Mr. *Putnam's* method, to form his fraction will require over 8000 figures; after it is formed he must divide in full to 100 places, requiring 20000 figures. The contracted part of his division will require 10000 more figures; in all, upwards of 38000 figures. The work is, therefore, nearly double that by the common method.

PROBLEMS.

218.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A merchant marked his goods at an advance of 60 per cent. on their cost and then gave his customers "40 per cent. off", thinking that he was clearing a profit of 20 per cent. Did he gain or lose, and at what rate per cent.?

219.—Proposed by THOMAS BAGOT, County Superintendent of Ripley County, New Marion, Indiana.

A rope attached to the top of a vertical pole standing upon a plain just reaches to the ground. Desiring to find the hight of the pole I took hold of the end of the rope, keeping it perfectly taut, and pulled it out a feet from the foot of the pole; I then found that the end of the rope was b feet from the ground.

How high was the pole?

220.—From the SATURDAY EVENING POST, Philadelphia, Pa., June 1, 1872.

A fisherman began a cast net with 36 meshes. He enlarged it by adding 12 meshes more to the third row; the same to the sixth, ninth, and so on to each and every successive third row, termed "widen-rows". The last row, *being a "widen-row"*, contained 360 meshes. Required the entire number of meshes comprised within the net.

Enterprise, S. C. EGO GEO.

221.—Proposed by MISS ELIZA D. CRANE, Brooklyn, New York.

It is required to determine the greatest number of marbles one inch in diameter that can be packed in a cubical box whose inside dimensions are each one foot.

222.—Proposed by ELIJAH A. SQUIER, Le Grand, Marshall County, Iowa.

The top and bottom diameters of a flat-bottom kettle holding an ale gallon are in the proportion of 5 to 3, and its depth is 12 inches. Required the two diameters.

223.—Proposed by H. H. HOUGH, Principal Linden Female Seminary, Doylestown, Bucks County, Pennsylvania; and J. M. TAYLOR, Milton, Umatilla County, Oregon.

Find x and y, by quadratics, from the equations $x^2+y=11$, $x+y^2=7$.

224.—Proposed by CHRISTIAN HORNUNG, M. A., Professor of Mathematics, Heidleburg College, Tiffin, Seneca Co., O.

Let A and B be two given points in the diameter of a circle equally distant from the center C. In AB take any point P, so that PC $\times$ radius $=(AC)^2$. Let CP meet the circumference in D; draw AE and BE to any point E in the circumference, and take the arc DQ double of DE; join PQ. Prove that AE $\times$ BE $=$ PQ $\times$ radius.

225.—Proposed by D. C. JONES, Saint Charles, Butler County, Ohio.

The product of the length of a field by the width divided by the diagonal will give a quotient equal to the number of acres in it plus $4\frac{1}{2}$, and 4 times the diagonal divided by the difference between the length and width will give a quotient equal to one-sixth of the length. Find the dimensions of the field.

226.—Proposed by Professor W. P. CASEY, C. E., San Francisco, California.

Solve, by quadratics, the equations $x^2-y^2=a^2$, $x^3+3xy^2=b^3$.

227.—Proposed by Professor E. J. EDMUNDS, B. S., Principal of Academic School No. 3, New Orleans, Louisiana.

A triangle being given it is required to compare its area with that of the triangle formed by connecting the feet of the bisectors of its three angles.

228.—Proposed by FRANK ALBERT, Professor of Mathematics, Pennsylvania State Normal School, Millersville, Lancaster County, Pennsylvania.

A ball, radius r, is enclosed in a cubical box in which it just fits; there are also eight infinite series of balls, one in each corner. Each ball in these series touches the three adjacent sides of the box and the two balls next to it, except the first in each series which touches the large ball. Find (1) the radius of the nth ball in one of the series, (2) the sum of the volumes of the first n balls in one of the series and (3) the sum of the volumes of *all* the balls.

229.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

In finding the solid diagonals of a series of parallelopipeds, it was noticed that the length, breadth and thickness are consecutive numbers, and that in extracting the square root, if the sums were only a unit less every diagonal would be an integer. The sides of the first parallelopiped are 3, 4, 5 It is required to find general expressions for the sides of the nth solid, and compute the dimensions of the 100th solid.

230.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Give an expeditious method of approximating to the cube root of a quantity, and find by it the cube root of 2 to at least one hundred places of decimals.

Solutions of these problems should be received by February 1, 1881.

SENIOR DEPARTMENT.

Solutions of Problems Proposed in Nos. 1, 2 & 3.

114.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

To find two numbers such that their sum shall be a square, the sum of their squares a square, and if the cube of each be added to the square of the other the sums shall be equal.

II.—Solution by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Let x and y be the numbers; then $x+y=\square \ldots\ldots (1)$, $x^2+y^2=\square \ldots\ldots (2)$, and $x^3+y^2=y^3+x^2 \ldots\ldots (3)$.

From (3), $x^2+xy+y^2=x+y \ldots\ldots (4)$. Put $y=m^2-x$ and (1) is a square and (2) becomes

$$2x^2-2m^2x+m^4=\square \ldots\ldots (5) \quad \text{and (3)}, \quad x^2-m^2x+m^4=\square \ldots\ldots (6).$$

Put $x=m^2n$ and (5) becomes by reducing $2n^2-2n+1=\square=(pn\pm1)^2$;

whence $$n=\frac{2(p-1)}{p^2-2} \quad \text{and then} \quad x=\frac{2m^2(m-1)}{p^2-2}.$$

Substituting this value of x in (6) and reducing we get $m=\dfrac{p^2-2}{\sqrt{(p^4-2p^3+2p^2-4p+4)}}$.

Assuming the expression under the radical $=(p^2\pm p\pm2)^2$ and solving we get $p=\frac{8}{5}$ or $\frac{5}{4}$ with two other impracticable values. Then $m=\frac{7}{13}$, and $x=\frac{105}{169}$ and $m^2-x=-\frac{56}{169}$ are the numbers, both values of p giving the same result.

To obtain positive values, other values of p must be found. It is evident on inspection that $p=1$, or 2, but both these values give zero as one of the numbers. Take $p=q+1$, substitute it in the expression and solve it as indicated above and we have $q=-4$ or $\frac{1}{4}$, and $p=-3$ or $\frac{5}{4}$; but these values of p give the results already obtained.

Take now $p=q-3$, and we find $q=4$ or $\frac{120}{17}$, and $p=1$ or $\frac{69}{17}$. Using the last value of p we find

$$m=\frac{4183}{3637}, \quad x=\frac{1768\times4183}{(3637)^2}=\frac{7395544}{13227769}, \quad \text{and} \quad m^2-x=\frac{10101945}{13227769}.$$

III.—Solution by REUBEN DAVIS, Bradford, Stark County, Illinois; and WALTER SIVERLY, Oil City, Venango Co., Pa.

Let $x(x+y)$ and $y(x+y)$ be the required numbers; then must

$$x(x+y)+y(x+y) \text{ or } (x+y)^2=\square \ldots\ldots (1), \quad x^2(x+y)^2+y^2(x+y)^2=(x^2+y^2)(x+y)^2=\square \ldots\ldots (2),$$

$$x^3(x+y)^3+y^2(x+y)^2=x^2(x+y)^2+y^3(x+y)^3 \quad \text{or} \quad (x^3-y^3)(x+y)^3=(x^2-y^2)(x+y)^2.$$

Dividing the last equation by $(x-y)(x+y)^3$ it becomes $x^2+xy+y^2=1 \ldots\ldots (3)$.

The conditions will all be fulfilled if $x^2+y^2=\square \ldots\ldots (4)$, and $x^2+xy+y^2=1 \ldots\ldots (5)$.

Transposing (5), $x^2+y^2=1-xy \ldots\ldots (6)$, and $x^2+2xy+y^2=1+xy \ldots\ldots (7)$.

Let $1-xy=n^2$; then $xy=1-n^2$, and substituting in (7), $(x+y)^2=2-n^2 \ldots\ldots (8)$.

To make (8) a square, substitute $q+1$ for n and put the result, $1-2q-q^2, =(mq-1)^2$; then will

$$q=\frac{2(m-1)}{m^2+1}, \text{ and } n=q+1=\frac{m^2+2m-1}{m^2+1}, \quad 2-n^2=(x+y)^2=\left(\frac{m^2-2m-1}{m^2+1}\right)^2,$$

$$x^2+y^2=n^2=\left(\frac{m^2+2m-1}{m^2+1}\right)^2, \; xy=1-n^2=\frac{4m-4m^3}{(m^2+1)^2}, \; (x-y)^2=x^2+y^2-2xy=\frac{m^4+12m^3+2m^2-12m+1}{(m^2+1)^2}.$$

To make the last expression a square, let

$$x^4+12m^3+2m^2-12m+1=(m^2+6m-17)^2;$$

then $m=\frac{3}{2}$. Substituting $r+\frac{3}{2}$ for m we shall have to make

$$r^4+18r^3+\tfrac{139}{2}r^2+\tfrac{177}{2}r+\tfrac{529}{16}=\square.$$

Let $r^2+9r+\frac{23}{4}$ be the root of this square; then $r=-\frac{15}{23}$ and $r+\frac{3}{2}=\frac{39}{46}$. Substituting this value for m,

$$x^2+y^2=\left(\frac{2993}{3637}\right)^2, \; xy=\frac{4269720}{(3637)^2}, \; (x+y)^2=\left(\frac{4183}{3637}\right)^2, \; (x-y)^2=\left(\frac{647}{3637}\right)^2, \; x=\frac{2415}{3637}, \; y=\frac{1768}{3637},$$

$$x(x+y)=\frac{10101945}{13227769}, \qquad y(x+y)=\frac{7395544}{13227769}.$$

115.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

There is a series of parallelopipeds whose dimensions differ from a perfect cube by one unit in only one of the edges. In every case the solid diagonal is an integer. Calling the one whose edges are 1, 2, 2 the *first* parallelopiped, it is required to find general expressions for the dimensions of the nth solid, and compute the length, breadth and thickness of the 30th one.

II.—Solution by REUBEN DAVIS, Bradford, Stark County, Illinois.

Let x, x and $x-1$ be the edges of one of the parallelopipeds; then, representing the diagonal of the solid by D, we have $3x^2-2x+1=D^2$ ....(1).

Assume $3x^2-2x+1=(nx-1)^2=n^2x^2-2nx+1$; then $x=\dfrac{2(n-1)}{n^2-3}$. To make this value of x integral, n may be ± 2. If $n=2$, $x=2$; if $n=-2$, $x=-6$.

For the edges of the first parallelopiped we have $x=2$, $x=2$ and $x-1=1$; and for the second $x=-6$, $x=-6$ and $x-1=-7$. But as we use only the squares of these numbers in finding the solid diagonal, the signs may be disregarded after the edges are obtained.

Substituting $y+r$ for x in (1), it becomes $3y^2-2y+1+(6y-2)r+3r^2=\square$.
Put this $=[\sqrt{(3y^2-2y+1)}-sr]^2$, $=3y^2-2y+1-2\sqrt{(3y^2-2y+1)}sr+s^2r^2$, and we get

$$r=\frac{2s\sqrt{(3y^2-2y+1)}+6y-2}{s^2-3}.$$

If $s=2$, $r=4\sqrt{(3y^2-2y+1)}+6y-2$; but if $s=-2$, $r=-4\sqrt{(3y^2-2y+1)}+6y-2$;

$$\therefore\ x=y+r=\pm 4\sqrt{(3y^2-2y+1)}+7y-2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2).$$

Let $a=2=$ one of the equal edges of the first solid and $b=-6=$ one of the two equal edges of the second; then, if x_3, x_4, x_5, etc., denote one of the two equal edges of the 3d, 4th, 5th, etc., solids, we have

$$x_3=4\sqrt{(3a^2-2a+1)}+7a-2=c=24,\quad x_4=-4\sqrt{(3b^2-2b+1)}+7b-2=d=-88,$$

$$x_5=4\sqrt{(3c^2-2c+1)}+7c-2=e=330,\quad x_6=-4\sqrt{(3d^2-2d+1)}+7d-2=f=-1230,$$

$$x_7=4\sqrt{(3e^2-2e+1)}+7e-2=g=4592,\quad x_8=-4\sqrt{(3f^2-2f+1)}+7f-2=h=-17136,$$

$$x_9=4\sqrt{(3g^2-2g+1)}+7g-2=i=63954,\quad x_{10}=-4\sqrt{(3h^2-2h+1)}+7h-2=j=-238678,$$

$$x_{11}=4\sqrt{[3(x_8)^2-2x_8+1]}+7x_8-2,\ \ldots\ldots\ldots\ x_n=\pm\sqrt{[3(x_{n-2})^2-2x_{n-2}+1]}+7x_{n-2}-2.$$

In (1) substitute $z+w$ for x and it becomes $3(w+z)^2-2(w+z)+1=D^2$,

or

$$3z^2-2z+1+(6z-2)w+3w^2=\square.$$

Taking $(3z^2-2z+1)^{\frac{1}{2}}+\dfrac{(3z-1)w}{(3z^2-2z+1)^{\frac{1}{2}}}+\dfrac{w^2}{(3z^2-2z+1)^{\frac{3}{2}}}$ for the root, we find

$w=-2(3z-1)(3z^2-2z+1)=-(18z^3-18z^2+10z-2)$, and $x=w+z=-(18z^3-18z^2+9z-2)\ \ldots.\ (3)$.

Writing i for z in (3), we have $x_{28}=-(18i^3-18i^2+9i-2)=-4708351225665448=p$.

Substituting p for y in (2), $x_{30}=-4\sqrt{(3p^2-2p+1)}+7p-2=-65578872750585150$, and the edges of the 30th solid are 65578872750585150, 65578872750585150, 65578872750585151.

Substituting j for z in (3), we find $x_{31}=244743685008277952$, and this substituted for y in (2) gives us $x_{33}=3408839784121828674$; substituting this for z in (3) we find

$$x_{100}=-713004506105244899421887701590040546362489937656289477528,$$

and the edges of the 100th solid are

713004506105244899421887701590040546362489937656289477528,

713004506105244899421887701590040546362489937656289477528,

713004506105244899421887701590040546362489937656289477529.

Substituting the value of x_{100} for z in (3), we find

$x_{301}=$ 652453144804951535011845201847042805470321734238620521885494668004688136908727960404008078081815833732939213199039629958935050601555098839300287381755249691615536030967002,

and therefore the edges of the 301st solid are

652453144804951535011845201847042805470321734238620521885494668004688136908727960404008078081815833732939213199039629958935050601555098839300287381755249691615536030967002,

652453144804951535011845201847042805470321734238620521885494668004688136908727960404008078081815833732939213199039629958935050601555098839300287381755249691615536030967002,

652453144804951535011845201847042805470321734238620521885494668004688136908727960404008078081815833732939213199039629958935050601555098839300287381755249691615536030967001.

To find the diagonals: From (1), $x = \frac{1}{3}[1 \pm \sqrt{(3D^2-2)}]$; $\therefore\ 3D^2-2 = \square$ .... (4).
In (4), substitute $v+t$ for D and it becomes $3v^2-2+6vt+3t^2 = \square$. Assuming $\sqrt{(3v^2-2)} - tu$ for the root of this square we find $t = \frac{2\sqrt{(3v^2-2)}u+6v}{u^2-3}$. If $u=2$, $t = 4\sqrt{(3v^2-2)}+6v$; but if $u=-2$, then $t = -4\sqrt{(3v^2-2)}+6v$; therefore $D = v+t = \pm 4\sqrt{(3v^2-2)}+7v$ .... (5).

If $v=-1$, $D=-3$, but if $v=+1$, $D=11$. Let $a'=3$ = diagonal of first solid, $b'=11$ = diagonal of second solid, D_3, D_4 D_5, etc., the third, fourth, fifth, etc., diagonals; then

$$D_3 = 4\sqrt{(3a'^2-2)}+7a' = 41 = c', \qquad D_4 = 4\sqrt{(3b'^2-2)}+7b' = 153 = d',$$
$$D_5 = 4\sqrt{(3c'^2-2)}+7c' = 571 = e', \qquad D_6 = 4\sqrt{(3d'^2-2)}+7d' = 2131 = f',$$
$$D_7 = 4\sqrt{(3e'^2-2)}+7e' = 7953 = g', \qquad D_8 = 4\sqrt{(3f'^2-2)}+7f' = 29681 = h',$$
$$D_9 = 4\sqrt{(3g'^2-2)}+7g' = 110771 = i', \qquad D_{10} = 4\sqrt{(3h'^2-2)}+7h' = 413403 = j',$$
$$D_{11} = 4\sqrt{[3(D_9)^2-2]}+7D_9, \ldots\ldots D_n = 4\sqrt{[3(D_{n-2})^2-2]}+7D_{n-2}.$$

Let $3v^2-2+6vt+3t^2 = \left[(3v^2-2)^{\frac{1}{2}} + \frac{3vt}{(3v^2-2)^{\frac{1}{2}}} - \frac{3t^2}{(3v^2-2)^{\frac{3}{2}}}\right]^2$. Expanding and reducing we get

$$t = 2v(3v^2-2), \text{ and } D = v+t = 6v^3-3v = 3v(2v^2-1) \quad\ldots\ldots\ldots\ldots (6).$$

Writing i' for v in (6), $D_{28} = 3i'(2i'^2-1) = 8155103542731753$; substituting D_{28} for v in (5), we have

$$D_{30} = 4\sqrt{[3(D_{28})^2-2]}+7D_{28} = 113585939507107651;$$ substituting j' for v in (6),

$D_{31} = 3j'(2j'^2-1) = 423908497265970753$; and then $D_{33} = 4\sqrt{[3(D_{31})^2-2]}+7D_{31} = 5904283700961130691$.

Writing the 33d diagonal for v in (6) we find the 100th diagonal,

$$D_{100} = 1234960030599837928682339736709998512373739432964939784153.$$

Using the last number in place of v in (6) we have

$D_{301} =$ 1130081996360269942983265922209550534263488386708182341826270548225312167556032490900564938293341593264019376761065450354578681963877363685857880056605581408120057091945003.

III.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Let x_n, x_n, $x_n \pm 1$ and y_n be the lengths of the respective edges and solid diagonal of the nth parallelopiped. The length of the solid diagonal is $\sqrt{[3(x_n)^2 \pm 2x_n + 1]}$; therefore $3(x_n)^2 \pm 2x_n + 1 = \square = (y_n)^2$, whence $9(x_n)^2 \pm 6x_n + 1 = \square = 3(y_n)^2 - 2 = (w_n)^2$ .... (1); and $3x_n \pm 1 = \sqrt{[3(y_n)^2-2]} = w_n$, whence $x_n = \frac{1}{3}(w_n \mp 1)$ .... (2). From (1), by transposition, $(w_n)^2 - 3(y_n)^2 = -2$ .... (3).

If we multiply together the equations $p^2-3q^2 = 1$, $t^2-3u^2 = -2$, we get $(pt+3qu)^2 - 3(pu+qt)^2 = -2$; hence we may take $w_n = pt+3qu$, $y_n = pu+qt$. But the least values of t and u are obviously $t=1$ and $u=1$; therefore $w = p+3q$, $y = p+q$.

The general values of p and q are $p_n = \frac{(2+\sqrt{3})^n + (2-\sqrt{3})^n}{2}$, $q_n = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}}$.

$\therefore w_n = \frac{1}{2}[(\sqrt{3}+1)(2+\sqrt{3})^n - (\sqrt{3}-1)(2-\sqrt{3})^n]$ and $y_n = \frac{1}{6}[(3+\sqrt{3})(2+\sqrt{3})^n + (3-\sqrt{3})(2-\sqrt{3})^n]$.

Substituting in (2) the edges of the nth solid are

$$\tfrac{1}{6}[(\sqrt{3}+1)(2+\sqrt{3})^n - (\sqrt{3}-1)(2-\sqrt{3})^n \mp 2], \quad \tfrac{1}{6}[(\sqrt{3}+1)(2+\sqrt{3})^n - (\sqrt{3}-1)(2-\sqrt{3})^n \mp 2]$$

and $$\tfrac{1}{6}[(\sqrt{3}+1)(2+\sqrt{3})^n - (\sqrt{3}-1)(2-\sqrt{3})^n \mp 2] \pm 1.$$

The upper sign obtains when n is even and the lower when n is odd.

Edges of second solid are 6, 6, 7; of third, 24, 24, 23; of fourth, 88, 88, 89; of fifth, 330, 330, 329; etc.

$$(2+\sqrt{3})^2 = 7+4\sqrt{3}, \qquad (2+\sqrt{3})^4 = 97+56\sqrt{3}, \qquad (2+\sqrt{3})^8 = 18817+10864\sqrt{3},$$
$$(2+\sqrt{3})^{10} = 262087+151316\sqrt{3}, \qquad (2+\sqrt{3})^{20} = 137379191137+79315912984\sqrt{3},$$
$$(2+\sqrt{3})^{30} = 72010600134783751+41575339372323900\sqrt{3};$$
$$(\sqrt{3}+1)(2+\sqrt{3})^{30} = 196736618251755451+113585939507107651\sqrt{3},$$
$$(\sqrt{3}-1)(2-\sqrt{3})^{30} = -196736618251755451+113585939507107651\sqrt{3}.$$

Substituting in the general formulas we find the edges of the 30th solid are

65578872750585150, 65578872750585150, 65578872750585151.

If we expand $\sqrt{3}$ by Continued Fractions the convergents are $\frac{p_1}{q_1} = \frac{1}{1}$, $\frac{p_2}{q_2} = \frac{2}{1}$, $\frac{p_3}{q_3} = \frac{5}{3}$, $\frac{p_4}{q_4} = \frac{7}{4}$, $\frac{p_5}{q_5} = \frac{19}{11}$, $\frac{p_6}{q_6} = \frac{26}{15}$, $\frac{p_7}{q_7} = \frac{71}{41}$, etc. The numerators and denominators of the odd convergents, $\frac{p_1}{q_1}$, $\frac{p_3}{q_3}$, $\frac{p_5}{q_5}$, etc., are the values of w and y satisfying (3). If $\frac{p_{2m+1}}{q_{2m+1}}$, $\frac{p_{2m+3}}{q_{2m+3}}$, $\frac{p_{2m+5}}{q_{2m+5}}$ be any three consecutive odd convergents, we have the relations $p_{2m+5} = 4p_{2m+3} - p_{2m+1}$ ..... (4), $q_{2m+5} = 4q_{2m+3} - q_{2m+1}$ ..... (5).

118.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.; and JOSEPH B. MOTT, Neosho, Mo.
Solve the equation $x^{x^x} = a$, and find the value of x when $a = 300$.

II.—Solution by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

Taking Naperian logarithms,

$$x^x \log x = \log a = b \quad \text{and} \quad x\log x + \log(\log x) = \log b = c.$$

Now put $\log x = 1+y$, whence $x = \varepsilon^{1+y}$ and $(1+y)\varepsilon^{1+y} + \log(1+y) = c$.

Developing, $(1+y)\varepsilon^{1+y} = 1+y+(1+y)^2+\frac{1}{2}(1+y)^3+\frac{1}{6}(1+y)^4+\frac{1}{24}(1+y)^5+\ldots\ldots$
and $\log(1+y) = y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y^5-\ldots\ldots$; and developing still further, and reducing to decimals, we have $ay+by^2+cy^3+dy^4+ey^5+fy^6+gy^7+\ldots\ldots = c-\varepsilon = \varphi$
where $a = +6.4365637$, $b = +3.5774227$, $c = +2.1455212$, $d = +0.3163087$, $e = +0.3025808$, $f = -0.1402390$. Reverting this series we have

$$y = \frac{\varphi}{a} - \frac{6\varphi^2}{a^3} + \frac{(2b^2-ac)\varphi^3}{a^5} - \frac{(a^2d-5abc+5b^3)\varphi^4}{a^7} + \frac{(14b^4-21ab^2c+6a^2bd+30a^2c^2-a^3e)\varphi^5}{a^9} - \ldots\ldots,$$

or $\log x = 1+y = 1+A\varphi+B\varphi^2+C\varphi^3+D\varphi^4+E\varphi^5+\ldots$ in which $A = +0.1553624$, $B = -0.0134155$, $C = +0.0010668$, $D = +0.0000109$, $E = -0.0000340$; or for logarithmic calculation we have

$$\log x = 1+[9.1913459]\varphi+[8.1276080_n]\varphi^2+[7.0281006]\varphi^3+[5.0378928]\varphi^4+[5.5321611_n]\varphi^5\ldots\ldots$$

When $x^{x^x} = 300$ we have Nap. $\log 300 = 5.7037825 = b$, Nap. $\log b = 1.7411295 = c$, $\varepsilon = 2.7182818284$, $\varphi = c-\varepsilon = -0.9771523$, $\log\varphi = 9.9899622_n$ and $x = 2.303511$.

120.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.
Prove that for all positive values of k less than unity the equation $(x+a)(x+b) = k(x+c)^2$ has two real roots.

III.—Solution by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Missouri.

Developing and reducing, we have $(1-k)x^2+(a+b-2kc)x+ab-kc^2 = 0$ .... (1).
Now we must prove that $(a+b-2kc)^2 > 4(1-k)(ab-kc^2)$ for all positive values of k less than unity. When $k=1$, we have $(a-b)^2+4k(a-c)(b-c) = (a+b-2c)^2$, a positive quantity. Hence for all positive values of k less than unity, we have $(a-b)^2+4k(a-c)(b-c) > 0$ .... (2).
Developing (2), adding $4ab-4kc^2-4kab+4k^2c^2$ to both members, we find

$$(a+b)^2-4kc(a+b)+4k^2c^2 > 4ab-4kc^2-4kab+4k^2c^2,$$

or

$$(a+b-2kc)^2 > 4(1-k)(ab-kc^2).$$

The second solution published in No. 4 of the VISITOR, p. 104, is, in my opinion, defective. In the next to the last line the "therefore $kc^2 > ab$" does not follow. Take $a=4$, $b=5$, $c=6$, and $k=\frac{1}{2}$; then the equation reduces to $x^2+6x+4=0$, whose roots are real, but the last term is positive. I understand that a, b, and c may have any values whatever.

Mr. *Nichols'* solution was selected for publication on account of its brevity, and the defect pointed out above by Prof. *Seitz* was not noticed. The solutions by Prof. *Wood* and Mr *Pollard* were similar in method, but not defective.

123.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.
To find three whole numbers such that the sum of the squares of any two of them increased by the product of the same two shall be a rational square.

II.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Let x, y and z be the numbers, and we must find

$$x^2+xy+y^2 = \square, \qquad x^2+xz+z^2 = \square, \qquad y^2+yz+z^2 = \square.$$

Put $x^2+xy+y^2 = \left(\frac{px}{q}-y\right)^2$, then $\frac{x}{y} = \frac{q^2+2pq}{p^2-q^2}$. Take $p=2$, $q=1$; then $\frac{x}{y} = \frac{5}{3}$. Now let $5m = x$, $3m = y$ and $wm = z$; then the other expressions become, after expunging m^2,

$$25+5w+w^2 = \square \ldots\ldots\ldots\ldots (1), \qquad 9+3w+w^2 = \square \ldots\ldots\ldots\ldots (2).$$

Put (1) $= (nw-5)^2$ and we obtain $w = \frac{5(2n+1)}{n^2-1}$. Substituting in (2) and reducing,

$$9n^4+30n^3+97n^2+70n+19 = \square \ldots\ldots\ldots\ldots (3).$$

Putting (3) $= (3n^2+5n+\frac{1}{2}a)^2$ we get $3(24-a)n^2+5(14-a)n = \frac{1}{4}a^2-19$ .... (4).

Let $b+19=a$, then (4) becomes $3(5-b)n^2-5(5+b)n=\frac{1}{4}(b^2+38b+285)$. Now take $b=0$ and we have tne quadratic $3n^2-5n=\frac{57}{4}$, which gives $n=\frac{19}{6}$; therefore $w=\frac{264}{65}$, and $z=\frac{264m}{65}$. Taking $m=65$ we have $x=325$, $y=195$, $z=264$.

III.—Solution by the PROPOSER.

Denote the numbers by x, y and z, and the conditions to be satisfied are

$$x^2+xy+y^2=\square\ldots\ldots(1),\qquad x^2+xz+z^2=\square\ldots\ldots(2),\qquad y^2+yz+z^2=\square\ldots\ldots(3).$$

If $x=m^2-n^2$ and $y=2mn+n^2$ (1) is a square, and if $z=\frac{(2pq+q^2)x}{p^2-q^2}$ (2) is a square. Substituting this value of z in (3), we have $y^2+\left(\frac{2pq+q^2}{p^2-q^2}\right)xy+\left(\frac{2pq+q^2}{p^2-q^2}\right)^2x^2=\square$.

Expanding the above, and arranging the terms according to the powers of p, etc., we have

$$y^2p^4+2xyp^3q+(4x^2+xy-2y^2)p^2q^2+(4x^2-2xy)pq^3+(x^2-xy+y^2)q^4=\square\ldots\ldots\ldots\ldots(4).$$

Let $(4)=\left(yp^2+xpq+\frac{(3x^2+xy-2y^2)q^2}{2y}\right)^2$; then, reducing, we have $p=-\frac{(3x+5y)q}{4y}$.

In order to get a positive value of p, let $p=t-\frac{(3x+5y)q}{4y}$; then by substitution in (4), etc., we have

$$y^2t^4-(xy+5y^2)qt^3+\tfrac{1}{8}(23x^2+38xy+59y^2)q^2t^2-\left(\frac{69x^3+75x^2y+99xy^2+45y^3}{16y}\right)q^3t$$
$$+\left(\frac{21x^2+18xy+9y^2}{16y}\right)^2q^4=\square,=\left(yt^2-\tfrac{1}{2}(x+5y)qt-\frac{(21x^2+18xy+9y^2)q^2}{16y}\right)^2.$$

Reducing we find $t=\frac{3q(5x^3+11x^2y+11xy^2+5y^3)}{2y(7x^2+6xy+3y^2)}$, whence $p=\frac{(9x^3+13x^2y+27xy^2+15y^3)q}{4y(7x^2+6xy+3y^2)}$;

$\therefore$ we may take $p=9x^3+13x^2y+27xy^2+15y^3$, and $q=4y(7x^2+6xy+3y^2)$.

Now, in the values of x, y m, n may be any numbers if $m>n$, whence p, q are known, and then z is found.

Let $m=2$, $n=1$; then $x=3$, $y=5$, $p=\frac{197q}{190}$; therefore $p=197$, $q=190$ and $z=\frac{110960}{903}$; and in integers, $x=2709$, $y=4515$, $z=110960$.

By this general method, any number of answers may be found, but they are all large numbers.

To find smaller numbers, let $m=2$, $n=1$; then $x=3$, $y=5$, whence (4) becomes $25p^4+30p^3q+p^2q^2+6pq^3+19q^4=\square\ldots\ldots\ldots(5)$. This is so when $p=\pm\frac{1}{2}q$, $\pm q$, $-3q$. Let $p=\frac{1}{2}q+r=\frac{1}{2}(2r+q)$; then by substitution, etc., we have from (5)

$$400r^4+1280r^3q+1336r^2q^2+672rq^3+441q^4=\square=(20r^2+32rq+21q^2)^2.$$

Reducing we find $r=-\frac{14q}{11}$, $p=-\frac{17q}{22}$; $\therefore$ $p=-17$, $q=22$, $z=\frac{264}{65}$; or, in integers, $x=195$, $y=325$, $z=264$.

The above expression may also be put $=\left(21q^2+16rq+\frac{180r^2}{7}\right)^2$, whence $r=\frac{7q}{4}$, $p=\frac{9q}{4}$; $\therefore$ $p=9$, $q=4$, $z=\frac{264}{65}$ as before.

Let $p=r-3q$, then, by substitution, the expression (5) becomes

$$25r^4-270r^3q+1081r^2q^2-1890rq^3+1225q^4=\square=(5r^2-27rq-35q^2)^2;$$

whence we get $r=\frac{70q}{13}$, $\therefore$ $p=\frac{31q}{13}$, $p=31$, $q=13$, $z=\frac{325}{88}$; or in integers, $x=264$, $y=440$, $z=325$.

Proceeding in a similar way, we may find other sets of numbers, and also by varying m and n.

A solution by *Reuben Davis* will be published in a future No.

125.—Proposed by ORLANDO D. OATHOUT, Read, Clayton County, Iowa.

What is the average thickness of a slab sawed at random from a round log?

II.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania; and ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.

Let BLC be the flat side of the slab, AL $= x =$ its thickness in the middle.

The average thickness $=$ sum of all the ordinates DE divided by the number of them. The sum of all the ordinates may be represented by the area of the cross-section ABC, and their number by the width BC of the slab.

BC $= 2\sqrt{(2rx-x^2)}$, r being the radius of the log, and the area ABC is easily found by the rules of Trigonometry to be $r^2\cos^{-1}\left(\frac{r-x}{r}\right)-(r-x)\sqrt{(2rx-x^2)}$; therefore, when the thickness at the middle is known, the average thickness of the slab is

$$\frac{r^2\cos^{-1}\left(\frac{r-x}{r}\right)-(r-x)\sqrt{(2rx-x^2)}}{2\sqrt{(2rx-x^2)}}.$$

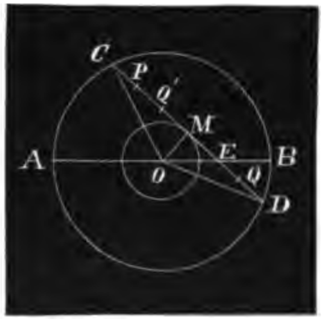
But when the slab is sawed at random from a given log the thickness x is not known, and may be anything from 0 to r. Hence the required average thickness is

$$\Delta=\frac{\int_0^r\left[r^2\cos^{-1}\left(\frac{r-x}{r}\right)-(r-x)\sqrt{(2rx-x^2)}\right]dx}{\int_0^r 2\sqrt{(2rx-x^2)}dx}, =\frac{4r}{3\pi}.$$

127.—Proposed by Samuel Roberts, M. A., F. R. S., Member of the London Mathematical Society, London, England.

Two random points being taken within a circle (1) on opposite sides of a given diameter, (2) on the same side, (3) anywhere; find in each case the average radius of the concentric circle touched by the chord through them.

II.—Solution by E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics North Missouri State Normal School, Kirksville, Missouri.



Let AB be the given diameter, O the center of the circle, P, Q, two random points on opposite sides of AB, P, Q′ two random points on the same side, CD the chord through P, Q, or P, Q′, and OM the radius of the concentric circle.

Let OA $= r$, DP $= x$, DQ or DQ′ $= y$, DE $= u$, $\angle$COM $= \theta$, and $\angle$EOM $= \varphi$. Then we have OM $= r\cos\theta$, CD $= 2r\sin\theta$ and

$$u = r(\sin\theta-\cos\theta\tan\varphi).$$

An element of the circle at P is $r\sin\theta\, d\theta dx$; at Q or Q′, $(x-y)d\varphi dy$.

1. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, $-\theta$ and θ, and doubled; of x, u and $2r\sin\theta = x'$; and of y, 0 and u.

Hence, since the whole number of ways the two points can be taken is $\frac{1}{4}\pi^2r^4$, the required average is

$$r_1=\frac{8}{\pi^2r^4}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}\int_u^{x'}\int_0^u r^2\sin\theta\cos\theta\, d\theta d\varphi dx(x-y)dy, =\frac{4}{\pi^2r^3}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}\int_u^{x'}[x^2-(x-u)^2]\sin\theta\cos\theta\, d\theta d\varphi dx,$$

$$=\frac{8r}{\pi^2}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}(1-\cos^2\theta\sec^2\varphi)\sin^2\theta\cos\theta\, d\theta d\varphi, =\frac{16r}{\pi^2}\int_0^{\frac12\pi}(\theta-\sin\theta\cos\theta)\sin^2\theta\cos\theta\, d\theta, =\frac{8r}{3\pi}-\frac{256r}{45\pi^2}.$$

2. The limits of θ are 0 and $\frac{1}{2}\pi$; from $\varphi=-\theta$ to $\varphi=\theta$, the limits of x are u and $2r\sin\theta$, and those of y are u and x, and doubled; and from $\varphi=\theta$ to $\varphi=\frac{1}{2}\pi$, the limits of x are 0 and $2r\sin\theta$, and those of y are 0 and x, and doubled. The result of the integration with respect to φ must be doubled.

Hence the required average is

$$r_2=\frac{16}{\pi^2r^4}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}\int_u^{x'}\int_u^x r^2\sin\theta\cos\theta\, d\theta d\varphi dx(x-y)dy+\frac{16}{\pi^2r^4}\int_0^{\frac12\pi}\int_\theta^{\frac12\pi}\int_0^{x'}\int_0^x r^2\sin\theta\cos\theta\, d\theta d\varphi dx(x-y)dy,$$

$$=\frac{8}{\pi^2r^3}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}\int_u^{x'}(x-u)^2\sin\theta\cos\theta\, d\theta d\varphi dx+\frac{8}{\pi^2r^3}\int_0^{\frac12\pi}\int_\theta^{\frac12\pi}\int_0^{x'}x^2\sin\theta\cos\theta\, d\theta d\varphi dx,$$

$$=\frac{8r}{3\pi^2}\int_0^{\frac12\pi}\int_{-\theta}^{+\theta}(\sin\theta+\cos\theta\tan\varphi)^3\sin\theta\cos\theta\, d\theta d\varphi+\frac{64r}{3\pi^2}\int_0^{\frac12\pi}\int_\theta^{\frac12\pi}\sin^4\theta\cos\theta\, d\theta d\varphi,$$

$$=\frac{16r}{3\pi^2}\int_0^{\frac12\pi}(2\pi\sin^2\theta-3\theta+3\sin\theta\cos\theta)\sin^2\theta\cos\theta\, d\theta, =\frac{256r}{45\pi^2}-\frac{8r}{15\pi}.$$

3. The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and 2π; of x, 0 and $2r\sin\theta = x'$; and of y, 0 and x, and doubled. Hence the required average is

$$r_3 = \frac{2}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{x'}\int_0^{x} r^3 \sin\theta\cos\theta\, d\theta d\varphi dx (x-y)dy, = \frac{1}{\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{x'} x^2 \sin\theta\cos\theta\, d\theta d\varphi dx,$$

$$= \frac{8r}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\sin^4\theta\cos\theta\, d\theta d\varphi, = \frac{16r}{3\pi}\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos\theta\, d\theta, = \frac{16r}{15\pi}.$$

III.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

1. Let O be the center of the circle, AB the fixed diameter, M, N the random points, CD the chord through them intersecting AB in K, OH the perpendicular on CD which is the radius of the concentric circle; OA $= r$, CM $= x$, DN $= y$, $\angle$COH $= \theta$, $\angle$KOH $= \varphi$, $\Delta =$ the required average.

An element at M $= r\sin\theta\, d\theta dx$, at N $= y dy d\varphi$. When H is above AB, CK $= r(\sin\theta + \cos\theta\tan\varphi) = u$,
DK $= r(\sin\theta - \cos\theta\tan\varphi) = v$;
when H is below AB, CK $= v$ and DK $= u$.

The limits of x are 0 and u when H is above AB, and 0 and v when H is below AB. The limits of y are 0 and v when H is above AB, and 0 and u when H is below. The limits of φ are 0 and θ and doubled, and the limits of θ are 0 and $\frac{1}{2}\pi$ and doubled.

Since the whole number of ways that the two points can be taken is $\frac{1}{4}\pi^2 r^4$,

$$\Delta = \frac{16}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left[\int_0^{u}\int_0^{v} r\cos\theta . r\sin\theta\, dx y dy + \int_0^{v}\int_0^{u} r\cos\theta . r\sin\theta\, dx y dy\right] d\theta d\varphi,$$

$$= \frac{8}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left[\int_0^{u} v^2\cos\theta\sin\theta\, dx + \int_0^{v} u^2\cos\theta\sin\theta\, dx\right] d\theta d\varphi, = \frac{8}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\cos\theta\sin\theta(uv^2 + vu^2)d\theta d\varphi,$$

$$= \frac{16r}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}(\sin^4\theta\cos\theta - \sin^2\theta\cos^3\theta\tan^2\varphi)d\theta d\varphi, = \frac{16r}{\pi^2}\int_0^{\frac{1}{2}\pi}(\theta\sin^2\theta\cos\theta - \sin^3\theta\cos^2\theta)d\theta, = \frac{8r}{3\pi} - \frac{256r}{45\pi^2}.$$

2. In this case let M, N′ or M′, N″ be the two random points, both above AB, CN′ or C′N″ $= x$, MN′ or M′N″ $= y$, C′D′ $= 2r\sin\theta = w$.

The limits of y are 0 and x; of x, 0 and v when H is below AB, and 0 and u when H is above and D below, and 0 and w when the chord C′D′ is wholly above AB; of φ, 0 and θ and doubled when the chord crosses AB, and θ and $\pi - \theta$ and *not* doubled for the chord C′D′; of θ, 0 and $\frac{1}{2}\pi$ and doubled.

$$\therefore\ \Delta_1 = \frac{16}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left[\int_0^{v}\int_0^{x} r\cos\theta . r\sin\theta\, dx y dy + \int_0^{u}\int_0^{x} r\cos\theta . r\sin\theta\, dx y dy\right] d\theta d\varphi$$

$$+ \frac{8}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_{\theta}^{\pi-\theta}\int_0^{w}\int_0^{x} r\cos\theta . r\sin\theta\, d\theta d\varphi dx y dy,$$

$$= \frac{8}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\left[\int_0^{v}\cos\theta\sin\theta x^2 dx + \int_0^{u}\cos\theta\sin\theta x^2 dx\right] d\theta d\varphi + \frac{4}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_0^{\pi-\theta}\int_0^{w}\cos\theta\sin\theta x^2 dx d\theta d\varphi,$$

$$= \frac{16r}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int^{\theta}(\sin^4\theta\cos\theta + 3\sin^2\theta\cos^3\theta\cot^2\varphi)d\theta d\varphi + \frac{32r}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\pi-\theta}\cos\theta\sin^4\theta\, d\theta d\varphi,$$

$$= \frac{16r}{3\pi^2}\int_0^{\frac{1}{2}\pi}(2\pi\sin^4\theta\cos\theta - 3\theta\sin^2\theta\cos\theta + 3\sin^3\theta\cos^2\theta)d\theta = \frac{256r}{45\pi^2} - \frac{8r}{15\pi}.$$

3. In this case the number of ways that the two points can be taken is $\pi^2 r^4$,

$$\therefore\ \Delta_2 = \frac{2}{\pi^2 r^4}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{w}\int_0^{x} r\cos\theta . r\sin\theta\, d\theta d\varphi dx y dy, = \frac{1}{\pi^2 r^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^{w}\cos\theta\sin\theta\, d\theta d\varphi x^2 dx,$$

$$= \frac{8r}{3\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{2\pi}\sin^4\theta\cos\theta\, d\theta d\varphi, = \frac{16r}{3\pi}\int_0^{\frac{1}{2}\pi}\sin^4\theta\cos\theta\, d\theta, = \frac{16r}{15\pi}.$$

129.—Proposed by REUBEN DAVIS, Bradford, Stark County, Illinois.

It is required to find three positive numbers, such that if each be diminished by the cube of their sum the three remainders will be rational cubes.

II.—Solution by the PROPOSER.

Let x, y and z denote the numbers; then must

$x-(x+y+z)^3 =$ a cube ....(1), $y-(x+y+z)^3 =$ a cube ....(2), $z-(x+y+z)^3 =$ a cube ....(3).

Put $x-(x+y+z)^3=\frac{m^3}{q^3}(x+y+z)^3$........(4), $y-(x+y+z)^3=\frac{n^3}{q^3}(x+y+z)^3$........(5),

$$z-(x+y+z)^3=\frac{p^3}{q^3}(x+y+z)^3 \text{........(6).}$$

By addition, $(x+y+z)-3(x+y+z)^3=\frac{m^3+n^3+p^3}{q^3}(x+y+z)^3$..... .(7), and if $m^3+n^3+p^3=q^3$ then $(x+y+z)-3(x+y+z)^3=(x+y+z)^3$, whence $x+y+z=\frac{1}{2}$. Substituting this value of $x+y+z$ in (4), (5), (6) we find $x=\frac{m^3+q^3}{8q^3}$, $y=\frac{n^3+q^3}{8q^3}$, $z=\frac{p^3+q^3}{8q^3}$.

If $m=3$, $n=4$, $p=5$, then $q=6$ and $x=\frac{243}{1728}$, $y=\frac{280}{1728}$, $z=\frac{341}{1728}$.

From (7) we obtain $(x+y+z)^2\left(\frac{m^3+n^3+p^3}{q^3}+3\right)=1$. Take $m=3$, $n=5$, $p=6$, $q=2$; then $49(x+y+z)^2=1$, and $x+y+z=\frac{1}{7}$.

$$\therefore x=\left(\frac{m^3+q^3}{q^3}\right)\times\frac{1}{7^3}=\frac{5}{392},\quad y=\left(\frac{n^3+q^3}{q^3}\right)\times\frac{1}{7^3}=\frac{19}{392},\quad z=\left(\frac{p^3+q^3}{q^3}\right)\times\frac{1}{7^3}=\frac{32}{392}.$$

III.—Solution by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

Let $\frac{x^3+1}{a^3}$, $\frac{y^3+1}{a^3}$, $\frac{z^3+1}{a^3}$ be the numbers, and put the sum of the numerators,

$x^3+y^3+z^3, =\square, =a^2$......(1); then the sum of the numbers $=\frac{a^2}{a^3}=\frac{1}{a}$, the cube of which, subtracted from each number, gives the cube remainders $\frac{x^3}{a^3}$, $\frac{y^3}{a^3}$, $\frac{z^3}{a^3}$.

Let $x=mw-1$, $y=nw-1$, $z=pw+2$; then, substituting in (1), we have

$$(m^3+n^3+p^3)w^3-3(m^2+n^2-2p^2)w^2+3(m+n+4p)w+9=\square=[3+\tfrac{1}{2}(m+n+4p)w]^2;$$

this being reduced we have $w=\frac{(m+n+4p)^2+12(m^2+n^2-2p^2)}{4(m^3+n^3+p^3)}$, in which expression m, n must be taken unequal and positive, and $p=0$, or any number that will make z, if negative, <1, w being positive.

Let $m=1$, $n=3$, $p=-2$; then $w=\frac{1}{2}$, $x=-\frac{1}{2}$, $y=\frac{1}{2}$, $z=1$, and $a=2$; whence the numbers are $\frac{7}{64}$, $\frac{9}{64}$, $\frac{16}{64}$.

Let $m=2$, $n=4$, $p=-3$; then $w=\frac{1}{3}$, $x=-\frac{1}{3}$, $y=\frac{1}{3}$, $z=1$, and $a=2$; whence the numbers are $\frac{13}{108}$, $\frac{14}{108}$, $\frac{27}{108}$.

These two answers are the least possible.

If $m=0$, then $x=-1$ whatever the value of w, and we shall have two numbers answering the conditions.

Let $m=0$, $n=4$, $p=-1$; then $w=\frac{2}{3}$, $x=-1$, $y=\frac{5}{3}$, $z=\frac{4}{3}$, $a=3$; whence the numbers are $\frac{152}{729}$, $\frac{91}{729}$.

131.—Proposed by DE VOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

A spherical homogeneous mass m, radius r, contracts by the mutual attraction of its particles to a radius nr; if the work thus expended be suddenly changed into heat, how many degrees F. will the temperature of the mass be increased, its specific heat being s and the heat uniformly disseminated?

II.—Solution by the PROPOSER.

Let $g'=$ the acceleration of gravity on sphere m, then, *initially*, $g'\frac{\rho}{r}=$ the acceleration of any particle at distance ρ from the center. *During* contraction the force varies as the inverse squares; hence if x be any distance between ρ and $n\rho$, we have $\frac{d^2x}{dt^2} = -g'\frac{\rho}{r}\cdot\frac{\rho^2}{x^2}$, which integrated between the limits ρ and $n\rho$ gives $\frac{1}{2}v^2 = g'\frac{\rho^2}{r}\cdot\frac{1-n}{n}$. The mass of a spherical shell will be $dm = 3m\frac{\rho^2 d\rho}{r^3}$, and the work of contracting the sphere will be $$\tfrac{1}{2}\int v^2dm = 3mg'\frac{1-n}{r^4n}\int_0^r \rho^4 d\rho = \tfrac{3}{5}mrg'\frac{1-n}{n}.$$

To find g', consider the earth a homogeneous sphere, its mass M, radius R, gravity at the surface g; then $g' = \frac{m}{M}\cdot\frac{R^2}{r^2}g$. Let $m = \mu M$, $r = yR$, $J =$ Joule's mechanical equivalent of heat, $\delta =$ density of the earth compared with water. Substituting the value of g', dividing by J and also by the weight of a sphere of water equal in volume to that of the given sphere $\left(\text{or } \frac{\mu Mg}{\delta}\right)$, the result will be the number of degrees F. that the temperature would be increased if the body was composed of water; hence for the given body we have $t_0 = \frac{3}{5}\frac{\delta\mu R}{Jsy}\cdot\frac{1-n}{n}$. Let $R = 20893000$ feet, the mean radius of the earth, $J = 772$ foot-pounds, $\delta = 5\frac{1}{2}$, and we have $$t_0 = 89310\frac{\mu}{sy}\cdot\frac{1-n}{n}.$$

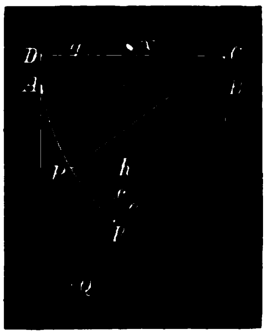
Remark.—It is not necessary, as assumed above, that the particles fall freely; the result would be the same if they met with resistances, but in either case the initial and terminal velocities must be the same in order that the work done by the attraction through the space considered shall *all* be changed into heat.

As a practical example, suppose the given body be the earth, contracted from double its present diameter to its present volume, and that its mean specific heat be $\frac{1}{8}$ (or the same as that of iron); then $\mu = 1$, $y = 2$, $s = \frac{1}{8}$, $n = \frac{1}{2}$, and we have $t_0 = 357240°$ F.

133.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A duck swims across a river a rods wide, always aiming for a point in the bank b rods up stream from a point opposite the place she started from. The velocity of the current is v miles an hour, and the duck can swim n miles an hour in still water. Required the equation of the *curve* the duck describes in space, and the distance she swims in crossing the river.

III.—Solution by Enoch Beery Seitz, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Adair County, Missouri.



Let A be the point from which the duck starts, B the point directly opposite A, C the point for which the duck continually aims, P and p two consecutive positions of the duck while crossing the river. Draw PQ parallel to CB, meeting Cp produced in Q, and draw Pe perpendicular to CQ. Then PQ is the distance the duck would be carried by the water in the time of swimming from P to p, and Qp is the distance she would swim in still water.

Let AB $= a$, BC $= b$, AC $= c$, CP $= r$, $\angle$PCD $= \theta$. Then Pe $= rd\theta$, $pe = dr$, Qe = Pe$\tan\theta = r\tan\theta\, d\theta$, PQ = Pe$\sec\theta = r\sec\theta\, d\theta$, Q$p = \frac{n}{v}$.PQ $= \frac{nr}{v}\sec\theta\, d\theta$. But $pe =$ Qe − Qp; therefore $dr = r\tan\theta\, d\theta - \frac{nr}{v}\sec\theta\, d\theta$.

Dividing by r, we have $\frac{dr}{r} = \tan\theta\, d\theta - \frac{n}{v}\sec\theta\, d\theta$.

Integrating, and observing that when $r = a$, $\cos\theta = \frac{a}{c}$, we find

$$\log\left(\frac{r\cos\theta}{a}\right) = \frac{n}{v}\log\left[\left(\frac{b+c}{a}\right)\left(\frac{\cos\theta}{1+\sin\theta}\right)\right], \text{ or } r\cos\theta = a\left[\left(\frac{b+c}{a}\right)\left(\frac{\cos\theta}{1+\sin\theta}\right)\right]^{\frac{n}{v}},$$

the polar equation to the curve.

Passing to rectangular co-ordinates, taking CD for the axis of x, and putting $\frac{v}{n} = m$, we have

$$2y = (b+c)\left(\frac{x}{a}\right)^{1-m} - \frac{a^2}{b+c}\left(\frac{x}{a}\right)^{1+m}.$$

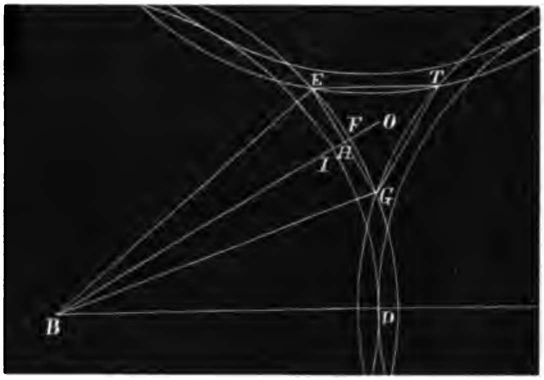
The distance the duck swims is

$$\int\sqrt{(dx^2+dy^2)} = \tfrac{1}{2}\int_0^a\left[(1-m)^2\left(\frac{b+c}{a}\right)^2\left(\frac{x}{a}\right)^{-2m} + 2(1+m^2) + (1+m)^2\left(\frac{a}{b+c}\right)^2\left(\frac{x}{a}\right)^{2m}\right]^{\frac{1}{2}}dx.$$

135.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Three equal circles touch each other externally; find the average area of all the circles that can be drawn in the space enclosed by them.

II.—Solution by the PROPOSER.



Let B be the center of one of the equal given circles, $a=$ its radius, $=\mathrm{BD}=\mathrm{BI}$, $x=\mathrm{IF}=$ radius of one of the circles drawn in the space enclosed by the given circles. The number of circles, radius x, that can be drawn in this space is represented by the curvilinear equilateral triangle ETG.

Let $\theta=\angle\mathrm{FBG}=\angle\mathrm{FBE}$, $\varphi=\angle\mathrm{GBD}$, $r=\mathrm{BF}=a+x$, $u=$ area of the curvilinear triangle ETG, and $\Delta=$ the average area required.

Then $\mathrm{EG}=2r\sin\theta$, area of the rectilinear equilateral triangle $\mathrm{ETG}=r^2\sqrt{3}\sin^2\theta$, sector $\mathrm{EBG}=r^2\theta$, segment $\mathrm{EFGH}=r^2\theta-r^2\sin\theta\cos\theta$.

$\therefore\quad u=r^2\sqrt{3}\sin^2\theta+3r^2\sin\theta\cos\theta-3r^2\theta\ \ldots\ldots(1).$

Also, $\varphi+\theta=\frac{1}{6}\pi\ldots(2)$. But $\cos\varphi=\dfrac{a}{r}\ldots(3)$;

therefore $\quad\varphi=\cos^{-1}\left(\dfrac{a}{r}\right)$, and from (2), $\quad\theta=\frac{1}{6}\pi-\varphi,=\frac{1}{6}\pi-\cos^{-1}\left(\dfrac{a}{r}\right)\ldots\ldots\ldots\ldots(4).$

$$\sin\theta=\sin(\tfrac{1}{6}\pi-\varphi),=\tfrac{1}{2}\cos\varphi-\tfrac{1}{2}\sqrt{3}\sin\varphi,=\frac{a-[\sqrt{(r^2-a^2)}]\sqrt{3}}{2r}\ \ldots\ldots\ldots\ldots(5),$$

$$\cos\theta=\cos(\tfrac{1}{6}\pi-\varphi),=\tfrac{1}{2}\sqrt{3}\cos\varphi+\tfrac{1}{2}\sin\varphi,=\frac{a\sqrt{3}+\sqrt{(r^2-a^2)}}{2r}\ \ldots\ldots\ldots\ldots(6).$$

Substituting in (1) the values of θ, $\sin\theta$ and $\cos\theta$ obtained from (4), (5) and (6), we have

$$u=\tfrac{1}{2}\left[2a^2\sqrt{3}-\pi r^2-6a\sqrt{(r^2-a^2)}+6r^2\cos^{-1}\left(\frac{a}{r}\right)\right].$$

The limits of r are $r=\mathrm{BI}=a$, and $r=\mathrm{BO}=\frac{2}{3}a\sqrt{3},=r'$, O being the center of the circle inscribed in the rectilineal equilateral triangle ETG.

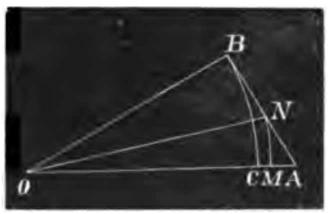
$$\therefore\quad\Delta=\frac{\int_a^{r'}\pi(r-a)^2u\,dr}{\int_a^{r'}u\,dr}=\frac{\pi\int_a^{r'}\left[2a^2\sqrt{3}-\pi r^2-6a\sqrt{(r^2-a^2)}+6r^2\cos^{-1}\left(\frac{a}{r}\right)\right](r-a)^2dr}{\int_a^{r'}\left[2a^2\sqrt{3}-\pi r^2-6a\sqrt{(r^2-a^2)}+6r^2\cos^{-1}\left(\frac{a}{r}\right)\right]dr}.$$

$$\int_a^{r'}\left[2a^2\sqrt{3}-\pi r^2-6a\sqrt{(r^2-a^2)}+6r^2\cos^{-1}\left(\frac{a}{r}\right)\right](r-a)^2dr=\tfrac{1}{180}a^5(6\pi+308-320\sqrt{3}+207\log3),$$

$$\int_a^{r'}\left[2a^2\sqrt{3}-\pi r^2-6a^2\sqrt{(r^2-a^2)}+6r^2\cos^{-1}\left(\frac{a}{r}\right)\right]dr=\tfrac{1}{3}a^3(\pi+4-6\sqrt{3}+3\log3);$$

$$\therefore\quad\Delta=\tfrac{1}{60}\pi a^2\left(\frac{6\pi+308-320\sqrt{3}+207\log3}{\pi+4-6\sqrt{3}+3\log3}\right).$$

III.—Solution by LUCIUS BROWN, Hudson, Middlesex County, Massachusetts.



In the figure O is the center of one of the given circles, and A is the center of the enclosed space. ABC is one-sixth of that space, and is similar to each of the other five-sixths.

Let $\mathrm{OC}=r$, $\mathrm{OM}=x$; then arc $\mathrm{MN}=x\left[\frac{1}{6}\pi-\cos^{-1}\left(\frac{r}{x}\right)\right]$.

Let any point in MN be taken as the center of a circle (radius ρ) drawn in the enclosed space. Then we have for the limits of x, r and $\frac{2}{3}r\sqrt{3}$ $(=r_1)$, and for those of ρ, 0 and $x-r$ $(=\rho_1)$.

Hence the average area of the variable circle is

$$\frac{\pi\int_r^{r_1}\int_0^{\rho_1}x\left[\frac{1}{6}\pi-\cos^{-1}\left(\frac{r}{x}\right)\right]\rho^2dx\,d\rho}{\int_r^{r_1}\int_0^{\rho_1}x\left[\frac{1}{6}\pi-\cos^{-1}\left(\frac{r}{x}\right)\right]dx\,d\rho}=\frac{\pi\int_r^{r_1}\frac{1}{3}x(x-r)^3\left[\frac{1}{6}\pi-\cos^{-1}\left(\frac{r}{x}\right)\right]dx}{\int_r^{r_1}x(x-r)\left[\frac{1}{6}\pi-\cos^{-1}\left(\frac{r}{x}\right)\right]dx}$$

$$=\tfrac{1}{60}\pi r^2\left(\frac{6\pi+308-320\sqrt{3}+207\log3}{\pi+4-6\sqrt{3}+3\log3}\right).$$

137.—Proposed by E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Adair County, Missouri.

Two points are taken at random in the surface of a circle, but on opposite sides of a given diameter; find (1) the chance that the chord drawn through them does not exceed a line of given length, and (2) the average length of the chord.

II.—Solution by Walter Siverly, Oil City, Venango County, Pennsylvania.

1. Let O be the center of the circle, AOB the fixed diameter, M, N or m, n the random points, CD or cd the chord through them intersecting AB in K or k, OH or Oh the perpendicular on CD or cd; OA $= r$, CM or $cm = x$, DM or $dm = y$, $\angle$COH or $\angle c$O$h = \theta$, $\angle$KOH or $\angle k$O$h = \varphi$; $2r\sin\beta$ the given length of the chord, $p =$ required probability, $\Delta =$ the required average.

An element of the circle at M or $m = r\sin\theta\, d\theta dx$, at N or $n = ydyd\varphi$;

$CK = r(\sin\theta + \cos\theta\tan\varphi) = u, \quad ck = r(\sin\theta - \cos\theta\tan\varphi) = v.$

The limits of x are 0 and u for CM and 0 and v for cm; of y, 0 and v for DN and 0 and u for dn; of φ, 0 and θ and doubled; of θ, 0 and β and doubled for p, and 0 and $\frac{1}{2}\pi$ and doubled for Δ.

Since the whole number of ways the two points can be taken is $\frac{1}{4}\pi^2r^4$,

$$p = \frac{16}{\pi^2r^4}\int_0^\beta\int_0^\theta\left[\int_0^u\int_0^v r\sin\theta\,dxydy + \int_0^v\int_0^u r\sin\theta\,dxydy\right]d\theta d\varphi,$$

$$= \frac{8}{\pi^2r^3}\int_0^\beta\int_0^\theta\left[\int_0^u v^2\sin\theta\,dx + \int_0^v u^2\sin\theta\,dx\right]d\theta d\varphi, = \frac{8}{\pi^2r^3}\int_0^\beta\int_0^\theta(uv^2+vu^2)\sin\theta\,d\theta d\varphi,$$

$$= \frac{16}{\pi^2}\int_0^\beta\int_0^\theta(\sin^4\theta - \sin^2\theta\cos^2\theta\tan^2\varphi)d\theta d\varphi = \frac{16}{\pi^2}\int_0^\beta(\theta\sin^3\theta - \sin^3\theta\cos\theta)d\theta,$$

$$= \frac{16}{\pi^2}[\tfrac{1}{4}\beta^2 - \tfrac{1}{2}\beta\sin\beta\cos\beta + \tfrac{1}{4}\sin^2\beta(1-\sin^2\beta)] = \left(\frac{2\beta-\sin2\beta}{\pi}\right)^2.$$

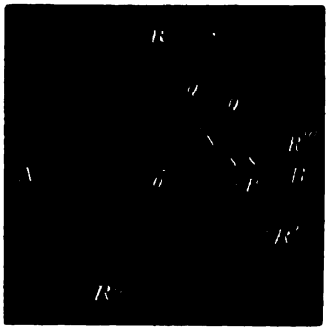
Cor.—When the given line is equal to the radius of the circle, $2\beta = 60°, = \frac{1}{3}\pi$, and $p = \left(\frac{1}{3} - \frac{\sqrt{3}}{2\pi}\right)^2$.

2. $$\Delta = \frac{16}{\pi^2r^4}\int_0^{\frac{1}{2}\pi}\int_0^\theta\left[\int_0^u\int_0^v 2r\sin\theta.r\sin\theta\,dxydy + \int_0^v\int_0^u 2r\sin\theta.r\sin\theta\,dxydy\right]d\theta d\varphi,$$

$$= \frac{16}{\pi^2r^2}\int_0^{\frac{1}{2}\pi}\int_0^\theta\left[\int_0^u v^2\sin^2\theta\,dx + \int_0^v u^2\sin^2\theta\,dx\right]d\theta d\varphi, = \frac{16}{\pi^2r^2}\int_0^{\frac{1}{2}\pi}\int_0^\theta\sin^2\theta(uv^2+vu^2)d\theta d\varphi,$$

$$= \frac{32r}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^\theta(\sin^5\theta - \sin^3\theta\cos^2\theta\tan^2\varphi)d\theta d\varphi, = \frac{32r}{\pi^2}\int_0^{\frac{1}{2}\pi}(\theta\sin^3\theta - \sin^4\theta\cos\theta)d\theta, = \frac{832r}{45\pi^2}.$$

III.—Solution by Samuel Roberts, M. A., F. R. S., Member of the London Mathematical Society, London, England.



1. Let O be the center and AOB the given diameter of the circle ARBR′. Take a random point P in the lower semicircle and through it draw two chords RPR′, R″PR‴ equal to the given line and meeting the diameter in S, S′ respectively.

If S′ lies between S and B, the favorable space for the other random point Q is BSR — BS′R‴; if otherwise the favorable space is BSR.

Put $\angle$BOP $= \varphi$, $\angle$OPR $= \frac{1}{2}\pi - \theta$, $\angle$ROR′ $= 2\alpha$, OP $= \rho$, OB $= r$, $\rho' = r\cos\alpha$. Then if p is the required chance

$$\tfrac{1}{4}\pi^2r^4p = 2\int_{\rho'}^r\left[\int_0^{\alpha+\theta}(\mathrm{BSR})d\varphi - \int_0^{\alpha-\theta}(\mathrm{BS'R'''})d\varphi\right]\rho d\rho,$$

$$= r^2\int_{\rho'}^r\int_0^{\alpha+\theta}[\alpha+\theta-\varphi-\tfrac{1}{2}\sin2\alpha+\cos^2\alpha\tan(\varphi-\theta)]d\varphi\rho d\rho$$

$$- r^2\int_{\rho'}^r\int_0^{\alpha-\theta}[\alpha-\theta-\varphi-\tfrac{1}{2}\sin2\alpha+\cos^2\alpha\tan(\varphi+\theta)]d\varphi\rho d\rho,$$

$$= r^2(2\alpha-\sin2\alpha)\int_{\rho'}^r\theta\rho d\rho, = \tfrac{1}{4}r^4(2\alpha-\sin2\alpha)^2, \text{ and } p = \left(\frac{2\alpha-\sin2\alpha}{\pi}\right)^2.$$

2. Put OS $= x$, SQ $= y$, SP $= z$, SR $= y_1$, SR$' = z_1$, $\angle$BSQ $= \theta$ and let Δ be the required average. The element at Q is $\sin\theta\, dxdy$ and that at P is $(y+z)dzd\theta$. Then we have

$$\Delta = \frac{4}{\pi^2 r^4}\int_0^{\pi}\int_{-r}^{+r}\int_0^{y_1}\int_0^{z_1}(y_1+z_1)(y+z)\sin\theta\, d\theta dx dy dz, = \frac{2}{\pi^2 r^4}\int_0^{\pi}\int_{-r}^{+r} y_1 z_1 (y_1+z_1)^2 \sin\theta\, d\theta dx.$$

But y_1, $-z_1$ are the roots of $y^2 + 2xy\cos\theta - (r^2 - x^2) = 0$, and therefore

$$\Delta = \frac{8}{\pi^2 r^4}\int_0^{\pi}\int_{-r}^{+r}[(r^2-x^2)^2 + (r^2-x^2)x^2\cos^2\theta]\sin\theta\, d\theta, = \frac{16}{\pi^2 r^4}\int_{-r}^{+r}[(r^2-x^2)^2 + \tfrac{1}{3}(r^2-x^2)x^2]dx, = \frac{832r}{45\pi^2}.$$

138.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Two sides of a plane triangle are a and b; find its average area.

I.—Solution by the PROPOSER.

Let $x =$ third side, $\Delta =$ average area required; then $\frac{1}{4}\sqrt{[(a+b)^2 - x^2]}\sqrt{[x^2 - (a-b)^2]} =$ area of the triangle, $= A$, and $\Delta = \int_{a-b}^{a+b} A dx \div \int_{a-b}^{a+b} dx, = \frac{1}{8b}\int_{a-b}^{a+b}\sqrt{[(a+b)^2 - x^2]}\sqrt{[x^2 - (a-b)^2]}dx.$

Let $(a+b)^2 - x^2 = y^2$; then $x = \sqrt{[(a+b)^2 - y^2]}$, $dx = \frac{-ydy}{\sqrt{[(a+b)^2 - y^2]}}$; the limits are $y = 0$, $y = 2\sqrt{(ab)} = y'$, and
$$\Delta = \frac{1}{8b}\int_0^{y'}\frac{y^2\sqrt{(4ab - y^2)}dy}{\sqrt{[(a+b)^2 - y^2]}}.$$

Let $\frac{4ab}{(a+b)^2} = e^2$, and $y = z\sqrt{(4ab)}$, then $\Delta = \frac{2a^2b}{a+b}\int_0^1\frac{z^2\sqrt{(1-z^2)}dz}{\sqrt{(1-e^2z^2)}}$.

$$\frac{z^2\sqrt{(1-z^2)}}{\sqrt{(1-e^2z^2)}} = \frac{1}{e^2}\cdot\frac{z^2\sqrt{(1-e^2z^2)}}{\sqrt{(1-z^2)}} - \frac{1-e^2}{e^2}\cdot\frac{z^2}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}},$$

$$\therefore\ \Delta = \tfrac{1}{2}a(a+b)\int_0^1\frac{z^2\sqrt{(1-e^2z^2)}dz}{\sqrt{(1-z^2)}} - \frac{a(a-b)^2}{2(a+b)}\int_0^1\frac{z^2dz}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}} \quad\ldots\ldots\ldots\ (1).$$

$$\frac{z^2\sqrt{(1-e^2z^2)}}{\sqrt{(1-z^2)}} = \frac{2e^2-1}{3e^2}\cdot\frac{\sqrt{(1-e^2z^2)}}{\sqrt{(1-z^2)}} - \tfrac{1}{3}\cdot\frac{1-2(1+e^2)z^2+3e^2z^4}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}} + \frac{1-e^2}{3e^2}\cdot\frac{1}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}},$$

$$\frac{z^2}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}} = -\frac{1}{e^2}\cdot\frac{\sqrt{(1-e^2z^2)}}{\sqrt{(1-z^2)}} + \frac{1}{e^2}\cdot\frac{1}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}};$$

$$\therefore\ \int\frac{z^2\sqrt{(1-e^2z^2)}dz}{\sqrt{(1-z^2)}} = \tfrac{1}{3}\left(\frac{2e^2-1}{e^2}\right)\mathrm{E}(e, z) - \tfrac{1}{3}z\sqrt{(1-z^2)}\sqrt{1-e^2z^2}) + \tfrac{1}{3}\left(\frac{1-e^2}{e^2}\right)\mathrm{F}(e, z),$$

and
$$\int\frac{z^2dz}{\sqrt{(1-z^2)}\sqrt{(1-e^2z^2)}} = -\frac{1}{e^2}\mathrm{E}(e, z) + \frac{1}{e^2}\mathrm{F}(e, z).$$

Substituting in (1) we finally obtain

$$\Delta = \frac{a+b}{12b}\left[(a^2+b^2)\mathrm{E}\left(\frac{2\sqrt{(ab)}}{a+b}\right) - (a-b)^2\mathrm{F}\left(\frac{2\sqrt{(ab)}}{a+b}\right)\right].$$

II.—Solution by HENRY HEATON, B. S., Atlantic, Case County, Iowa.

Let $x =$ the third side. Then the area of any triangle is $\frac{1}{4}\sqrt{[(a+b)^2 - x^2]}\sqrt{[x^2 - (a-b)^2]}, = A$, and the average area, $\Delta = \int_{a-b}^{a+b} A dx \div \int_{a-b}^{a+b} dx, = \frac{1}{8b}\int_{a-b}^{a+b}\sqrt{[(a+b)^2 - x^2]}\sqrt{[x^2 - (a-b)^2]}dx.$

Put $(a+b)^2 - x^2 = 4ab\sin^2\theta$, and $\frac{4ab}{(a+b)^2} = e^2$. Then $\Delta = \frac{2a^2b}{a+b}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta\, d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$.

Put $\frac{\sin^2\theta\cos^2\theta\, d\theta}{\sqrt{(1-e^2\sin^2\theta)}} = m\sqrt{(1-e^2\sin^2\theta)}d\theta + \frac{nd\theta}{\sqrt{(1-e^2\sin^2\theta)}} + pd[\sin\theta\cos\theta\sqrt{(1-e^2\sin^2\theta)}].$

Then $\sin^2\theta\cos^2\theta = m(1-e^2\sin^2\theta) + n + p[1 - 2(1+e^2)\sin^2\theta + 3e^2\sin^4\theta]$; whence, $m+n+p = 0$, $e^2m + 2(1+e^2)p = -1$ and $3e^2p = -1$. $\therefore\ m = \frac{2-e^2}{3e^4}$, $n = -\frac{2(1-e^2)}{3e^4}$ and $p = -\frac{1}{3e^2}$.

$$\therefore\ \Delta = \frac{a+b}{12b}[(a^2+b^2)\mathrm{E}(e, \tfrac{1}{2}\pi) - (a-b)^2\mathrm{F}(e, \tfrac{1}{2}\pi)].$$

III.—Solution by CHARLES H. KUMMELL, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.

Let $x=$ third side, $\Delta_0=$ average area required; then if $\triangle=$ area of triangle (a, b, x) we have

$$\triangle = \tfrac{1}{4}\sqrt{[(a+b)^2-x^2]}\sqrt{[x^2-(a-b)^2]} \quad\ldots\ldots(1),$$

and the average area

$$\Delta_0=\int_{a-b}^{a+b}\triangle dx \div \int_{a-b}^{a+b}dx, = \frac{1}{8b}\int_{a-b}^{a+b}4\triangle dx, = \frac{1}{8b}\int_{a-b}^{a+b}\Big[-(a^2-b^2)^2+2(a^2+b^2)x^2-x^4\Big]\frac{dx}{4\triangle}\ldots(2).$$

We have

$$\Big[x.4\triangle\Big]_{a-b}^{a+b}=0=\int_{a-b}^{a+b}\Big(4\triangle+\frac{2x^2(a^2+b^2-x^2)}{4\triangle}\Big)dx=\int_{a-b}^{a+b}\Big[-(a^2-b^2)^2+4(a^2+b^2)x^2-3x^4\Big]\frac{dx}{4\triangle};$$

$$\therefore\ \int_{a-b}^{a+b}\frac{x^4dx}{4\triangle}=-\tfrac{1}{3}(a^2-b^2)^2\int_{a-b}^{a+b}\frac{dx}{4\triangle}+\tfrac{4}{3}(a^2+b^2)\int_{a-b}^{a+b}\frac{x^2dx}{4dx}\ldots\ldots(3).$$

Substituting this into (2) we obtain

$$\Delta_0=\frac{1}{12b}\int_{a-b}^{a+b}\Big[-(a^2-b^2)^2+(a^2+b^2)x^2\Big]\frac{dx}{4\triangle}\ldots\ldots(4).$$

Place $x=-b\cos\varphi+\sqrt{(a^2-b^2\sin^2\varphi)}$ ($\varphi=$ exterior angle included by b and x); then

$$dx=b\sin\varphi d\varphi-\frac{b^2\sin\varphi\cos\varphi d\varphi}{\sqrt{(a^2-b^2\sin^2\varphi)}}=\frac{b\sin\varphi d\varphi}{\sqrt{(a^2-b^2\sin^2\varphi)}}\Big[\sqrt{(a^2-b^2\sin^2\varphi)}-b\cos\varphi\Big]=\frac{bx\sin\varphi d\varphi}{\sqrt{(a^2-b^2\sin^2\varphi)}},$$

$4\triangle=2bx\sin\varphi$. We have now, the limits of φ being 0 and π,

$$\Delta_0=\frac{1}{12b}\int_0^\pi\Big[-(a^2-b^2)^2+(a^2+b^2)[b^2(1-2\sin^2\varphi)+a^2-2b\cos\varphi\sqrt{(a^2-b^2\sin^2\varphi)}]\Big]\frac{d\varphi}{2\sqrt{(a^2-b^2\sin^2\varphi)}},$$

$$=\frac{1}{12b}\int_0^\pi\Big[\frac{-a^2(a^2-b^2)+(a^2+b^2)(a^2-b^2\sin^2\varphi)}{\sqrt{(a^2-b^2\sin^2\varphi)}}\Big]d\varphi=\frac{a}{12b}\Big[(a^2+b^2)E_0^\pi\Big(\frac{b}{a}\Big)-(a^2-b^2)F_0^\pi\Big(\frac{b}{a}\Big)\Big]\ (5).$$

Place $x=\sqrt{(a^2+b^2+2ab\cos2\varphi')}, =\sqrt{[(a+b)^2-4ab\sin^2\varphi']}$ ($2\varphi'=$ exterior angle included by a and b); then $dx=-\dfrac{4ab\sin\varphi'\cos\varphi' d\varphi'}{\sqrt{[(a+b)^2-4ab\sin^2\varphi']}}$, $4\triangle=4ab\sin\varphi'\cos\varphi'$, and since the limits are $\frac{1}{2}\pi$ and 0 we have, inverting the limits in (4),

$$\Delta_0=\tfrac{1}{12b}\Big[(a^2+b^2)\int_0^{\frac{1}{2}\pi}\sqrt{[(a+b)^2-4ab\sin^2\varphi']}d\varphi'-(a-b)^2\int_0^{\frac{1}{2}\pi}\frac{d\varphi'}{\sqrt{[(a+b)^2-4ab\sin^2\varphi']}}\Big],$$

$$=\frac{a+b}{12b}\Big[(a^2+b^2)E_0^{\frac{1}{2}\pi}\Big(\frac{2\sqrt{(ab)}}{a+b}\Big)-(a-b)^2F_0^{\frac{1}{2}\pi}\Big(\frac{2\sqrt{(ab)}}{a+b}\Big)\Big]\ldots\ldots(6).$$

The two expressions (5), (6) are connected by the so-called Landen's substitution. (See *Analyst*, vol. v, pages 18, 19 and 97—100.)

139.—Proposed by M. H. DOOLITTLE, U. S. Coast Survey Office, Washington, D. C.

Suppose infinitesimal aerolites equally distributed through all space, everywhere moving equally in all directions with a given uniform and constant absolute velocity. The aggregate mass intercepted in a given time by a given stationary sphere is supposed to be known. Determine the effect upon the eccentricity of a spherical planet of given mass and volume moving in an eccentric orbit, all of whose elements are given.

Solution by DE VOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

If the velocity of the planet be less than that of the aerolites, the same mass will be intercepted as if the planet was at rest. Consider this case.

The change of the elements of the orbit will be due to two causes. 1st. The increase of the mass of the planet will increase the attractive force between the sun and planet. 2nd. The aerolites will cause a direct resistance to the motion of the planet.

Let M be the mass of the sun, m' the initial mass of the planet, s the distance between them, k a constant; then will the acceleration of one body in reference to the other at the end of time t (the time in the problem being unity) be $\dfrac{k(M+m'+mt)}{s^2}$, hence at distance unity $\mu=k(M+m'+mt)$, and $d\mu=kmdt$. For an elliptical orbit, we have from Mechanics $V_0^2r_0^2=\mu a(1-e^2)=c$, a being the semi-transverse axis. Since the changes are small, consider two quantities only to vary contemporaneously. If e be constant, we find $da=-\dfrac{kmc}{\mu^2(1-e^2)}dt$; therefore $\Delta e=-\dfrac{kmct}{\mu^2(1-e^2)}$ nearly.

Similarly, if a be constant we find $de = \frac{km(1-e^2)}{\mu}dt$; therefore $\Delta e = \frac{kmct}{\sqrt{[a\mu^3(a\mu-c)]}}$ nearly.

Hence, the major axis decreases and the eccentricity increases with the time, and the amount of change for one revolution may be found by making t equal to the corresponding time.

2d. The law of resistance is not given. The aerolites being infinitesimal, we do not consider the impact as between finite masses, but that they constitute a medium through which the planet moves. Considering the medium as of uniform density, D, and the resistance as varying with the square of the velocity, the case comes within one discussed by La Place, *Mecanique Celeste*, [8925], [8926].

The equations will become for this case $da = -\frac{2KDa^2}{\mu}d\theta$, $\frac{de}{e} = -\frac{KDa}{\mu}d\theta$; K being a constant depending upon the form and mass of the planet. If $\frac{KD}{\mu} = K'$, $a = \frac{2K'a_1^2\theta}{1+2K'a_1\theta}$, $e = \frac{e_1}{\sqrt{(1+2K'a_1\theta)}}$; a_1 and e_1 being initial. The major axis and eccentricity both decrease as the vectorial angle increases, and the orbit becomes more nearly circular.

The plane of the orbit will not be changed; and finally, the longitude of the perihelion will not be changed, *Mecanique Celeste*, [8916].

140.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A circle is drawn intersecting a given circle, its center being at a given distance from that of the given circle. Find the average area common to both circles.

I.—Solution by Henry Heaton, B. S., Atlantic, Cass County, Iowa.

Let a = radius of given circle, r = radius of variable circle, b = distance between the centers and $2x$ = length of the chord joining the points of intersection.

Then $x = \frac{1}{2b}\sqrt{(2a^2b^2+2a^2r^2+2b^2r^2-a^4-b^4-r^4)}$, and the area common to the two circles is

$$r^2\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right)+a^2\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right)-bx.$$

If $b < a$, the average area is

$$\Delta = \frac{\int_{a-b}^{a+b}\left[r^2\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right)+a^2\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right)-bx\right]dr}{\int_{a-b}^{a+b}dr}$$

Integrating by parts,

$$\Delta = \frac{\pi}{6b}\left[3a^2(a+b)-(a-b)^3\right]+\frac{1}{12b^2}\int_{a-b}^{a+b}\frac{[r^2(a^2-b^2+r^2)+6a^2r^2-6b^2x^2]dr}{x}.$$

Put $r^2 = (a+b)^2-4ab\sin^2\theta$, and $\frac{2\sqrt{(ab)}}{a+b}=e$; then $dr = -\frac{2e\sqrt{(ab)}\sin\theta\cos\theta\, d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$ and $x = 2a\sin\theta\cos\theta$.

$$\therefore\ \Delta = \frac{\pi}{6b}\left[2a^3+6a^2b-3ab^2+b^3\right]$$
$$+\frac{1}{3b(a+b)}\int_0^{\frac{1}{2}\pi}\left[\frac{a(b-2a)(a+b)^2+2ab(3a^2-10ab-b^2)\sin^2\theta+20a^2b^2\sin^4\theta}{\sqrt{(1-e^2\sin^2\theta)}}\right]d\theta,$$

$$= \frac{\pi}{6b}\left[2a^3+6a^2b-3ab^2+b^3\right]+\frac{a+b}{9b}\left[(a-b)^2\mathrm{F}(e,\tfrac{1}{2}\pi)-(7a^2+b^2)\mathrm{E}(e,\tfrac{1}{2}\pi)\right].$$

If $b > a$, the lower limit of integration should be $b-a$, and

$$\Delta = \tfrac{1}{2}\pi a(a+b)+\frac{a+b}{9a}\left[(b-a)^2\mathrm{F}(e,\tfrac{1}{2}\pi)-(7a^2+b^2)\mathrm{E}(e,\tfrac{1}{2}\pi)\right].$$

If $b = a$, then $\Delta = a^2(\pi - \frac{16}{9})$.

II.—Solution by Charles H. Kummell, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.

Let a = radius of given circle, r = radius of variable circle, b = distance between their centers; then we have the area common to both circles, whether $b > a$, $b = a$ or $b < a$,

$$A = a^2\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + r^2\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right) - 2\triangle \quad\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $\triangle$ = area of triangle (b, a, r),

$$= \tfrac14\sqrt{[(b+a)^2-r^2]}\sqrt{[r^2-(b-a)^2]}, = \tfrac14\sqrt{[-(b^2-a^2)^2+2(b^2+a^2)r^2-r^4]} \quad\ldots\ldots\ldots\ldots (2).$$

We have then the average area common to both circles

$$A_0 = \frac{\int_{r_1}^{r'}\left[a^2\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + r^2\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right) - 2\triangle\right]dr}{\int_{r_1}^{r'}dr}$$

$$= \frac{1}{r'-r_1}\int_{r_1}^{r'}\left[a^2\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + r^2\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right) - 2\triangle\right]dr \quad\ldots\ldots\ldots (3),$$

where $r' = b+a$ and $r_1 = b-a$ if $b > a$; $r' = 2a$, $r_1 = 0$ if $b = a$; $r' = a+b$ and $r_1 = a-b$ if $b < a$.

By partial integration we obtain

$$A_0 = \frac{1}{r'-r_1}\left[a^2r\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + \tfrac13 r^3\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right)\right]_{r_1}^{r'}$$
$$- \frac{1}{r'-r_1}\int_{r_1}^{r'}\left[a^2r\cdot\frac{2r}{4\triangle} + 2\triangle - \tfrac13 r^3\left(\frac{a^2-b^2+r^2}{r.4\triangle}\right)\right]dr,$$

$$= \frac{1}{r'-r_1}\left[a^2r\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + \tfrac13 r^3\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right)\right]_{r_1}^{r'}$$
$$- \frac{1}{r'-r_1}\int_{r_1}^{r'}\left[-\tfrac12(b^2-a^2)^2 + \tfrac43(2a^2+b^2)r^2 - \tfrac56 r^4\right]\frac{dr}{4\triangle} \quad\ldots\ldots\ldots (4).$$

We have $\left[r.4\triangle\right]_{r_1}^{r'} = 0 = \int_{r_1}^{r'}\left[4\triangle + \frac{2r^2(a^2+b^2-r^2)}{4\triangle}\right]dr = \int_{r_1}^{r'}\left[-(b^2-a^2)^2+4(b^2+a^2)r^2-3r^4\right]\frac{dr}{4\triangle}$;

$$\therefore \int_{r_1}^{r'}\frac{r^4dr}{4\triangle} = -\tfrac13(b^2-a^2)^2\int_{r_1}^{r'}\frac{dr}{4\triangle} + \tfrac43(b^2+a^2)\int_{r_1}^{r'}\frac{r^2dr}{4\triangle}.$$

Substituting this into (4), and placing $C = \frac{1}{r'-r_1}\left[a^2r\cos^{-1}\left(\frac{a^2+b^2-r^2}{2ab}\right) + \tfrac13 r^3\cos^{-1}\left(\frac{b^2-a^2+r^2}{2br}\right)\right]_{r_1}^{r'}$, (reserving the determination of its special values in the three cases for the present) we have

$$A_0 = C + \frac{1}{r'-r_1}\left[\tfrac29(b^2-a^2)^2\int_{r_1}^{r'}\frac{dr}{4\triangle} - \tfrac29(7a^2+b^2)\int_{r_1}^{r'}\frac{r^2dr}{4\triangle}\right] \quad\ldots\ldots\ \ldots\ldots (5).$$

Place $r = \sqrt{(a^2+b^2+2ab\cos2\varphi)}$ (2φ = exterior angle of the triangle b, a, r at the center of the given circle) or $r = \sqrt{[(a+b)^2-4ab\sin^2\varphi]}$; then $dr = -\frac{4ab\sin\varphi\cos\varphi d\varphi}{\sqrt{[(a+b)^2-4ab\sin^2\varphi)]}}$,

$4\triangle = \sqrt{[(a+b)^2-r^2]}\sqrt{[r^2-(a-b)^2]} = \sqrt{(4ab)}\sin\varphi.\sqrt{(4ab)}\cos\varphi = 4ab\sin\varphi\cos\varphi$, and the new limits are $\varphi' = 0$, $\varphi_1 = \frac12\pi$ for each of the three cases. Therefore in (5), after inverting the limits,

$$A_0 = C + \frac{2}{9(r'-r_1)}\left[(b^2-a^2)^2\int_0^{\frac12\pi}\frac{d\varphi}{\sqrt{[(a+b)^2-4ab\sin^2\varphi]}} - (7a^2+b^2)\int_0^{\frac12\pi}\sqrt{[(a+b)^2-4ab\sin^2\varphi]}d\varphi\right];$$

hence if $b > a$,

$$A_0 = \tfrac12 a(a+b)\pi + \frac{a+b}{9a}\left[(b-a)^2\mathbf{F}_0^{\frac12\pi}\left(\frac{2\sqrt{(ab)}}{a+b}\right) - (7a^2+b^2)\mathbf{E}_0^{\frac12\pi}\left(\frac{2\sqrt{(ab)}}{a+b}\right)\right].$$

If $b = a$, $A_0 = \pi a^2 - \frac{16}{9}a^2\mathbf{E}_0^{\frac12\pi}(1) = a^2(\pi - \frac{16}{9})$.

If $b < a$,

$$A_0 = \frac{\pi}{6b}\left[3a^2(a+b) - (a-b)^3\right] + \frac{a+b}{9b}\left[(a-b)^2\mathbf{F}_0^{\frac12\pi}\left(\frac{2\sqrt{(ab)}}{a+b}\right) - (7a^2+b^2)\mathbf{E}_0^{\frac12\pi}\left(\frac{2\sqrt{(ab)}}{a+b}\right)\right].$$

141.—Proposed by J. J. SYLVESTER, LL. D., F. R. S., Corresponding Member of the Institute of France, Professor of Mathematics, Johns Hopkins University, Baltimore, Maryland.

A row of particles in contact are charged some with positive and some with negative electricity. Under the effect of their mutual actions or the influence of some external body, these electricities are subject to *continuous* variation so conditioned that when the electricity of any intermediate particle becomes neutral the electricities of the particles on either side of it are of opposite signs.

Supposing the electricities of each of the particles to be known for one moment of time and for some subsequent moment, show that a certain logical inference can be drawn as to the joint changes undergone by any two extreme or any other two particles in the interval, and state what it is.

I.—Solution by the PROPOSER.

The number of variations of sign in passing from particle to particle is to be constant for the one moment of time and for the other: the absolute numerical value of the difference between these two numbers will be equal or inferior to the sum of the number of changes of sign undergone by the electricities of the two extreme particles, and will differ from it by zero or an even number; for the change of sign of any intermediate particle can not alter the number of variations in the chain of particles, seeing that it can not take place by hypothesis without the two neighboring particles having at the moment of such change opposite signs: the total change brought about in the number of variations of sign between contiguous particles implies therefore the previous occurrence of at least that number of changes of sign at the extremities of the chain since it is only when one or the other of these undergoes a change of sign that a change in the total number of the variations of signs in the series can take place: that change may operate to increase or diminish the existing number by one unit, but a number n can not pass into the number by additions or subtractions of a *unit* without the unit being added or subtracted at least $\nu-n$ or $n-\nu$ times, whichever of these numbers is positive. Call the positive difference δ; the actual operations must evidently be δ of one kind μ of the same kind, and μ of the opposite kind; $\therefore$ $\delta+2\mu$ altogether, where μ is 0 or any positive integer.

A particular case of this, that in which one of the extreme particles is incapable of undergoing any change of sign, lies at the foundation of the proof of Sturm's Theorem.

The above is as purely a logical process of deduction as the so-called Quantification of the Predicate; and in my opinion the number of such forms of logical process is as far as we can say *a priori* unlimited.

Observe that into the solution of the problem there enters a pure act of Imagination, (that which consists in raising up the conception of counting the variations of sign from particle to particle,) such as Kant correctly observes takes place in the construction appertaining to any Geometrical problem or theorem; thus, ex. gr., in Euclid's proof of the exterior angle being equal to the sum of the interior and opposite angles of a triangle, the drawing of a line parallel to one of the sides is an exercise of the voluntary or imaginative and not of the reasoning faculty: in this act resides the real stress of the proof.

Perhaps according to this view it may be said that the greatest mathematician is not so much the closest reasoner as he who carries the most imagination into matters of the reason.

II.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let $+$ denote the positive, $-$ the negative and 0 the neutral particles, and suppose them to be arranged and change as in the following tables:—

0, +, 0, —, 0, +, 0, —, (1)	0, +, 0, —, 0, +, 0, —, (1)
+, 0, —, 0, +, 0, —, 0, (2)	—, 0, +, 0, —, 0, +, 0, (2)
0, —, 0, +, 0, —, 0, +, (3)	0, —, 0, +, 0, —, 0, +, (3)
—, 0, +, 0, —, 0, +, 0, (4)	+, 0, —, 0, +, 0, —, 0, (4)
0, +, 0, —, 0, +, 0, —, (5)	0, +, 0, —, 0, +, 0, —, (5)

Hence if the electricities in all the particles are the same we may know that each particle has passed through $4n$ changes.

If the neutrals are the same, and the positives and negatives occupy different positions, they have passed through $4n+2$ changes.

If the neutrals are changed to positive or negative we will know whether they have passed through $4n+1$ or $4n+3$ changes if we know which way the neutrals are changing in the given positions.

142.—Proposed by E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics North Missouri State Normal School, Kirksville, Missouri.

Two points are taken at random in the surface of an ellipse, one on each side of the major axis; find (1) the average distance between the points, and (2) the average length of the chord drawn through them.

Solution by the Proposer.

Let AA′ and BB′ be the axes of the ellipse, P, Q, the two random points, and RS the chord through them. Draw CD′ and AN parallel to RS, and CD conjugate to CD′, intersecting RS in M.

Let $CA=a$, $CB=b$, $CD=a'$, $CD'=b'$, $CM=x$, $CN'=x'$, $EP=y$, $EQ=z$, $RM=MS=u$, $EM=v$, $\angle REA'=\angle D'CA'=\theta$, $\angle DCA=\varphi$, and $\angle CMR=\psi$. Then we have $\sin\psi=\frac{ab}{a'b'}$, $x'=\frac{a\sin\theta}{\sin\psi}=\frac{a'b'\sin\theta}{b}$, $v=\frac{x\sin\varphi}{\sin\theta}$, $u^2=\frac{b'^2}{a'^2}(a'^2-x^2)$, $b'^2=\frac{b^2}{1-e^2\cos^2\theta}$, and $a'\sin\varphi=b'\surd(1-e^2)\cos\theta$.

An element of the ellipse at P is $\sin\psi\, dxdy$, and at Q it is $(y+z)d\theta dz$. The limits of θ are 0 and $\frac{1}{2}\pi$, and doubled; of x, $-x'$ and x'; of y, 0 and $u+v$; and of z, 0 and $u-v$. Hence, since the whole number of ways the two points can be taken is $\frac{1}{4}\pi^2a^2b^2$, the average distance between them is

$$\Delta=\frac{8}{\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}\int_0^{u+v}\int_0^{u-v}(y+z)^2\sin\psi\, d\theta dxdydz,$$

$$=\frac{8}{3\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}\int_0^{u+v}[(y+u-v)^3-y^3]\sin\psi\, d\theta dxdy,$$

$$=\frac{4}{3\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}(7u^4-6u^2v^2-v^4)\sin\psi\, d\theta dx,$$

$$=\frac{64b^2}{45\pi^2ae^2}\int_0^{\frac{1}{2}\pi}\left[\frac{9e^2-2}{(1-e^2\cos^2\theta)^2}+\frac{2(1-e^2)}{(1-e^2\cos^2\theta)^3}\right]\sin\theta\, d\theta,$$

$$=\frac{16a(15e^2+1)}{45\pi^2e^2}+\frac{8b^2(15e^2-1)}{45\pi^2ae^3}\log\left(\frac{1+e}{1-e}\right).$$

The average length of the chord through the points is

$$\Delta_1=\frac{8}{\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}\int_0^{u+v}\int_0^{u-v}2u(y+z)\sin\psi\, d\theta dxdydz,$$

$$=\frac{8}{\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}\int_0^{u+v}u[(y+u-v)^2-y^2]\sin\psi\, d\theta dxdy,$$

$$=\frac{16}{\pi^2a^2b^2}\int_0^{\frac{1}{2}\pi}\int_{-x'}^{+x'}u^2(u^2-v^2)\sin\psi\, d\theta dx,$$

$$=\frac{64b^2}{15\pi^2ae^2}\int_0^{\frac{1}{2}\pi}\left[\frac{5e^2-1}{(1-e^2\cos^2\theta)^2}+\frac{1-e^2}{(1-e^2\cos^2\theta)^3}\right]\sin\theta\, d\theta,$$

$$=\frac{8a(17e^2+1)}{15\pi^2e^2}+\frac{4b^2(17e^2-1)}{15\pi^2ae^3}\log\left(\frac{1+e}{1-e}\right).$$

Cor.—In the case of the circle, $e=0$ and the results reduce to $\Delta=\frac{1472a}{135\pi^2}$ and $\Delta_1=\frac{832a}{45\pi^2}$.

See solutions of Problems 29 and 137.

—143—

143.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A chord is drawn through two points taken at random in the surface of a circle; if a second chord be drawn through two other points taken at random in the surface, find the average area of the quadrilateral formed by joining the extremities of the chords.

I.—Solution by ENOCH BEERY SEITZ, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Adair County, Missouri.

Let AB be the chord through M, N, the first two random points, CD the chord through P, Q, the second two random points, O the center of the circle, and ABDC the quadrilateral formed. If the chords AB and CD intersect, the chords AC, AD, BC and BD will form the quadrilateral; and if C and D are both in the arc AEB, the chords AB, AD, CD and CB will form the quadrilateral. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $AM = w$, $MN = x$, $CP = y$, $PQ = z$, $AB = s$, $CD = v$, $\angle AOH = \theta$, $\angle COK = \varphi$, $\angle KOH = \psi$, and $\omega =$ the angle AB makes with some fixed line. Then $s = 2r\sin\theta$, $v = 2r\sin\varphi$; from $\psi = 0$ to $\psi = \theta - \varphi$ the area of the quadrilateral is

$$u_1 = \tfrac{1}{2}r^2[\sin 2\varphi - \sin 2\theta + 2\cos\psi\sin(\theta-\varphi)];$$

from $\psi = \theta - \varphi$ to $\psi = \theta + \varphi$ it is $u_2 = 2r^2\sin\theta\sin\varphi\sin\psi$; and from $\psi = \theta + \varphi$ to $\psi = \pi$ the area is $u_3 = \tfrac{1}{2}r^2[\sin 2\varphi + \sin 2\theta - 2\cos\psi\sin(\theta+\varphi)]$.

An element of the circle at M is $r\sin\theta\, d\theta dw$; at N, $xdxd\omega$; at P, $r\sin\varphi\, d\varphi dy$; and at Q, $zdzd\psi$.

The limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and doubled; of ψ, as stated above, and doubled; of ω, 0 and 2π; of w, 0 and s; of x, 0 and w, and doubled; of y, 0 and v; and of z, 0 and y, and doubled.

Hence, since the whole number of positions of the four points is π^4r^8, the required average is

$$\Delta = \frac{16}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}\left[\int_0^{\theta-\phi}u_1d\psi + \int_{\theta-\phi}^{\theta+\phi}u_2d\psi + \int_{\theta+\phi}^{\pi}u_3d\psi\right]\int_0^s\int_0^w\int_0^v\int_0^y r^2\sin\theta d\theta\sin\varphi d\varphi d\omega dw x dx dy z dz,$$

$$= \frac{8}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}\left[\int_0^{\theta-\phi}u_1d\psi + \int_{\theta-\phi}^{\theta+\phi}u_2d\psi + \int_{\theta+\phi}^{\pi}u_3d\psi\right]\int_0^s\int_0^w\int_0^v \sin\theta\sin\varphi d\theta d\varphi d\omega dw x dx y^2 dy,$$

$$= \frac{64}{3\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}\left[\int_0^{\theta-\phi}u_1d\psi + \int_{\theta-\phi}^{\theta+\phi}u_2d\psi + \int_{\theta+\phi}^{\pi}u_3d\psi\right]\int_0^s\int_0^w \sin^4\theta\sin\varphi d\theta d\varphi d\omega dw x dx,$$

$$= \frac{32}{3\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}\left[\int_0^{\theta-\phi}u_1d\psi + \int_{\theta-\phi}^{\theta+\phi}u_2d\psi + \int_{\theta+\phi}^{\pi}u_3d\psi\right]\int_0^s \sin^4\theta\sin\varphi d\theta d\varphi d\omega w^2 dw,$$

$$= \frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}\left[\int_0^{\theta-\phi}u_1d\psi + \int_{\theta-\phi}^{\theta+\phi}u_2d\psi + \int_{\theta+\phi}^{\pi}u_3d\psi\right]\sin^4\theta\sin^4\varphi d\theta d\varphi d\omega,$$

$$= \frac{256r^2}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^\theta\int_0^{2\pi}[(\pi-2\theta)\sin\theta\cos\theta + (\pi-2\varphi)\sin\varphi\cos\varphi + 2\sin^2\theta + 2\sin^2\varphi]\sin^4\theta\sin^4\varphi d\theta d\varphi d\omega,$$

$$= \frac{512r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^\theta[(\pi-2\theta)\sin\theta\cos\theta + (\pi-2\varphi)\sin\varphi\cos\varphi + 2\sin^2\theta + 2\sin^2\varphi]\sin^4\theta\sin^4\varphi d\theta d\varphi,$$

$$= \frac{32r^2}{81\pi^3}\int_0^{\frac{1}{2}\pi}[54(\pi\theta-2\theta^2)\sin\theta\cos\theta - 54\pi\sin^2\theta + 6(\pi-2\theta)(3\sin^4\theta + 10\sin^6\theta) + 105\theta$$

$$+ 216\theta\sin^2\theta - 105\sin\theta\cos\theta - 178\sin^3\theta\cos\theta - 128\sin^5\theta\cos\theta]\sin^4\theta d\theta, = \frac{35r^2}{9\pi}.$$

II.—Solution by Walter Siverly, **Oil City, Venango County, Pennsylvania.**

Let M, N be the first two random points, AB the chord through them, P, Q the second two random points, CD the chord through them and O the center of the circle. Draw OH and OK perpendicular to AB and CD.

Let $OA = r$, $AM = w$, $MN = x$, $CP = y$, $PQ = z$, $\angle AOH = \theta$, $\angle COK = \varphi$, $\angle AOC = \psi$ and $\mu =$ the angle AB makes with some fixed line. Then we have $AH = r\sin\theta$, and $CK = r\sin\varphi$.

An element of the circle at M is $r\sin\theta\, d\theta dw$; at N it is $d\mu dx$; at P it is $r\sin\varphi\, d\varphi dy$; and at Q, $d\psi z dz$.

When the chords intersect, the area of the quadrilateral is

$$\tfrac{1}{2}r^2[\sin\psi + \sin(2\varphi-\psi) + \sin(2\theta+\psi-2\varphi) - \sin(2\theta+\psi)] = u;$$

the limits of θ are 0 and $\frac{1}{2}\pi$; of φ, 0 and θ, and doubled; of ψ, 0 and 2φ, and doubled; of μ, 0 and 2π; of w, 0 and $2r\sin\theta = w'$; of x, 0 and w, and doubled; of y, 0 and $2r\sin\varphi = y'$; of z, 0 and y, and doubled.

When the chords do not intersect (see Fig. on preceding page), the area of the quadrilateral is

$$\tfrac{1}{2}r^2[\sin 2\varphi - \sin 2\theta + \sin(\psi-2\varphi) + \sin(2\theta-\psi)] = v;$$

the limits of θ are 0 and π; of φ, 0 and θ; of ψ, 2φ and 2θ; of μ, w, x, y, z, the same as above.

Hence, since the number of ways the four points can be taken is $\pi^4 r^8$, the required average area is

$$\Delta = \frac{16}{\pi^4 r^8}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'}\int_0^{y} ur\sin\theta\, d\theta\, r\sin\varphi\, d\varphi d\psi d\mu dw x dx dy z dz$$
$$+ \frac{4}{\pi^4 r^8}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'}\int_0^{y} vr\sin\theta\, d\theta\, r\sin\varphi\, d\varphi d\psi d\mu dw x dx dy z dz,$$

$$= \frac{8}{\pi^4 r^6}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'} u\sin\theta\sin\varphi\, d\theta d\varphi d\psi d\mu dw x dx y^2 dy$$
$$+ \frac{2}{\pi^4 r^6}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{w'}\int_0^{w}\int_0^{y'} v\sin\theta\sin\varphi\, d\theta d\varphi d\psi d\mu dw x dx y^2 dy,$$

$$= \frac{64}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'}\int_0^{w} u\sin\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu dw x dx$$
$$+ \frac{16}{3\pi^4 r^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{w'}\int_0^{w} v\sin\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu dw x dx,$$

$$= \frac{32}{3\pi^4 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi}\int_0^{w'} u\sin\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu w^2 dw$$
$$+ \frac{8}{3\pi^4 r^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi}\int_0^{w'} v\sin\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu w^2 dw,$$

$$= \frac{256}{9\pi^4}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{2\pi} u\sin^4\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu + \frac{64}{9\pi^4}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}\int_0^{2\pi} v\sin^4\theta\sin^4\varphi\, d\theta d\varphi d\psi d\mu,$$

$$= \frac{256r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}[\sin\psi + \sin(2\theta+\psi-2\varphi) + \sin(2\varphi-\psi) - \sin(2\theta+\psi)]\sin^4\theta\sin^4\varphi\, d\theta d\varphi d\psi$$
$$+ \frac{64r^2}{9\pi^3}\int_0^{\pi}\int_0^{\theta}\int_{2\phi}^{2\theta}[\sin 2\varphi - \sin 2\theta + \sin(\psi-2\varphi) + \sin(2\theta-\psi)]\sin^4\theta\sin^4\varphi\, d\theta d\varphi d\psi,$$

$$= \frac{2048r^2}{9\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\sin^6\theta\sin^6\varphi\, d\theta d\varphi + \frac{256r^2}{9\pi^3}\int_0^{\pi}\int_0^{\theta}[(\theta-\varphi)(\sin\varphi\cos\varphi - \sin\theta\cos\theta)$$
$$+ \sin^2\theta\cos^2\varphi - 2\sin\theta\cos\theta\sin\varphi\cos\varphi + \cos^2\theta\sin^2\varphi]\sin^4\theta\sin^4\varphi\, d\theta d\varphi.$$

$$= \frac{128r^2}{27\pi^3}\int_0^{\frac{1}{2}\pi}(15\theta - 8\sin^5\theta\cos\theta - 10\sin^3\theta\cos\theta - 15\sin\theta\cos\theta)\sin^6\theta\, d\theta + \frac{8r^2}{81\pi^3}\int_0^{\pi}(15\theta - 54\theta^2\sin\theta\cos\theta$$
$$+ 18\theta\sin^6\theta + 30\theta\cos^2\theta - 48\sin^7\theta\cos\theta + 2\sin^5\theta\cos\theta + 26\sin^3\theta\cos\theta - 15\sin\theta\cos\theta$$
$$+ 48\sin^4\theta\cos^3\theta - 12\sin^3\theta\cos\theta - 18\sin^5\theta\cos^3\theta)\sin^4\theta\, d\theta, = \frac{25r^2}{9\pi} + \frac{10r^2}{9\pi}, = \frac{35r^2}{9\pi}.$$

144.—Proposed by E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Missouri.

A circle is circumscribed about a triangle formed by joining three points taken at random in the surface of a circumscriptible polygon of n sides; (1) find the chance that the circle lies wholly within the polygon; and (2), a second circle being described in the same manner, find the chance that both circles lie wholly within the polygon and one of the circles is wholly within the other.

Solution by the Proposer.

1. Let ABCDEF.... be the polygon, PGR the triangle formed by joining the three random points P, G, R, O the center of the circle circumscribing PGR. Draw the polygon $abcdef$...., making its sides parallel to those of the given polygon, and at a distance from them equal to MP, and draw ONS and MH perpendicular to AB and PG.

Now while PG is given in length and direction, and the angle PRG is given, if MP is less than the radius of the inscribed circle of the given polygon, the area of the polygon $abcdef$.... represents the number of ways the three points can be taken, so that the circle circumscribing the triangle will lie wholly within the given polygon.

Let $PG=2x$, $OS=r$, perimeter of ABCDEF.... $=s$, area of segment $PRG=t$, area of sector $PMG=v$, area of triangle $PMG=u$, area of polygon ABCDEF.... $=\Delta$, $\angle PMH=\theta$, $\varphi=\sin^{-1}\left(\frac{x}{r}\right)$, and $\psi=$ the angle which PG makes with some fixed line.

Then we have $PM=x\operatorname{cosec}\theta$, $ON=r-x\operatorname{cosec}\theta$, area $abcdef.... =(r-x\operatorname{cosec}\theta)^2\frac{\Delta}{r^2}$, $v=\theta x^2\operatorname{cosec}^2\theta$, $u=x^2\cot\theta$, $t=v-u$, and $dt=dv-du=2x^2\operatorname{cosec}^2\theta(1-\theta\cot\theta)d\theta$.

An element of the polygon at G is $4xdxd\psi$, or $4r^2\sin\varphi\cos\varphi\, d\varphi d\psi$, and at R it is dt. The limits of x are 0 and r; of φ, 0 and $\frac{1}{2}\pi$; of θ, φ and $\pi-\varphi$; and of ψ, 0 and 2π.

Hence, doubling, since R may lie on either side of PG, we have for the required chance,

$$p=\frac{2}{\Delta^3}\int_0^r 4xdx\int_{\theta=\phi}^{\theta=\pi-\phi}dt\int_0^{2\pi}d\psi(r-x\operatorname{cosec}\theta)^2\frac{\Delta}{r^2},=\frac{16\pi}{r^2\Delta^2}\int_0^r xdx\int_{\theta=\phi}^{\theta=\pi-\phi}dt(r-x\operatorname{cosec}\theta)^2,$$

$$=\frac{32\pi}{r^2\Delta^2}\int_0^r x^3dx\int_\phi^{\pi-\phi}(r-x\operatorname{cosec}\theta)^2(1-\theta\cot\theta)\operatorname{cosec}^2\theta\, d\theta,$$

$$=\frac{2\pi r^4}{3\Delta^2}\int_0^{\frac{1}{2}\pi}[2(\pi-2\varphi)\sin 2\varphi+4-4\cos 4\varphi-3\sin^2 2\varphi\cos 2\varphi+64\sin^4\varphi\cos\varphi\log\tan\tfrac{1}{2}\varphi]d\varphi,$$

$$=\frac{2\pi^2r^4}{5\Delta^2},=\frac{8\pi^2r^2}{5s^2}.$$

Cor.—When the polygon becomes a circle this part of the problem is the same as Question 1843 in the *Educational Times*, $s=2\pi r$, and we have $p=\frac{2}{5}$, which agrees with the result of the Editor's solution on pp. 17 and 18, and Stephen Watson's on pp. 95—99, of vol. xii of the *Reprint*.

2. While the circle M is fixed, the number of ways the second circle can be described so that it will lie wholly within the circle M, is, according to the cor. of the first part of this solution, $\frac{2}{5}\pi^3x^6\operatorname{cosec}^6\theta$.

Hence, doubling, to allow for the number of ways the first circle may be wholly within the second, we have for the required chance

$$p_1=\frac{4}{\Delta^6}\int_0^r 4xdx\int_{\theta=\phi}^{\theta=\pi-\phi}dt\int_0^{2\pi}d\psi[\tfrac{2}{5}\pi^3x^6\operatorname{cosec}^6\theta(r-x\operatorname{cosec}\theta)^2]\frac{\Delta}{r^2},$$

$$=\frac{64\pi^4}{5r^2\Delta^5}\int_0^r x^7dx\int_{\theta=\phi}^{\theta=\pi-\phi}dt\operatorname{cosec}^6\theta(r-x\operatorname{cosec}\theta)^2,$$

$$=\frac{128\pi^4}{5r^2\Delta^5}\int_0^r x^9dx\int_\phi^{\pi-\phi}(r-x\operatorname{cosec}\theta)^2(1-\theta\cot\theta)\operatorname{cosec}^8\theta\, d\theta,$$

$$=\frac{16\pi^4r^{10}}{23625\Delta^5}\int_0^{\frac{1}{2}\pi}[105(\pi-2\varphi)\sin\varphi\cos\varphi+210\sin^2\varphi+170\sin^4\varphi+608\sin^6\varphi+6326\sin^8\varphi$$
$$+20334\sin^{10}\varphi-27648\sin^{12}\varphi+36750\sin^{10}\varphi\cos\varphi\log\tan\tfrac{1}{2}\varphi]d\varphi,$$

$$=\frac{8\pi^5r^{10}}{275\Delta^5},=\frac{256\pi^5r^5}{275s^5}.$$

Cor.—When the polygon becomes a circle, we have $p_1=\frac{8}{275}$.

145.—Proposed by BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Massachusetts.

Prove that if two bodies revolve about a center, acted upon by a force proportional to the distance from the center, and independent of the mass of the attracted body, each will appear to the other to move in a plane, whatever may be their mutual attraction.

I.—Quaternion Proof by the PROPOSER.

Let ρ and ρ_1 be the vectors of the two bodies referred to the center as origin.

Let $\rho' = D_t\rho,\ \rho_,' = D_t\rho_,,\ \rho'' = D_t\rho',\ \rho_,'' = D_t\rho_,'$, t being the time, and D the sign of differentiation.

If M is the central force at the unit of distance and N and $N_,$ the mutual attractions divided by the distance apart, we have

$$\rho'' = -M\rho + N(\rho_, - \rho),\quad \rho_,'' = -M\rho_, + N_,(\rho - \rho_,),\quad \rho_,'' - \rho'' = -M(\rho_, - \rho) + (N_, + N)(\rho - \rho'),$$

$$S[(\rho - \rho_,)(\rho' - \rho_,')(\rho'' - \rho_,'')] = (M + N + N_,)S(\rho - \rho_,)^2(\rho' - \rho_,') = 0$$

which proves that the apparent orbit is a plane.

II.—Solution by DE VOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, Hudson County, New Jersey.

Take the plane xy in the plane of the central force and the two bodies at any instant, the origin at the central force and the axis of x passing through the body a; the co-ordinates of a being x' and 0, and of the other body (b), x'' and y'', d their distance apart; M the central force at a unit's distance, F the force of b on a, and F' that of a on b (according to the Newtonian law $F = F'$); X', X'', Y', Y'' the axial accelerations of a and b.

Then $$X' = -Mx' + F\frac{x'' - x'}{d},\quad Y' = F\frac{x'}{d};\quad X'' = -Mx'' - F'\frac{x'' - x'}{d},\quad Y'' = -My'' - F'\frac{y''}{d};$$

$$\therefore\quad \frac{Y'' - Y'}{X'' - X'} = \frac{y''}{x'' - x'},$$

which gives the direction of the relative accelerations, and which is parallel to the line ab. Hence, whatever be the directions of motion of the two bodies or their absolute velocities, their relative positions (a, b), their relative velocities and their relative accelerations are parallel to a plane, which was to be proved.

Solved also by *Walter Siverly.*

146.—Proposed by Rev. W. J. WRIGHT, M. A., Ph. D., Member of the London Mathematical Society, Jenkintown, Pa.

If S_{ik} be the coefficient of a_{ik} in the determinant $D = \Sigma \pm a_{11}a_{22}\ldots\ldots a_{nn}$, and $\varDelta$ denote the determinant $\Sigma \pm S_{11}S_{22}\ldots\ldots S_{nn}$, prove that $\varDelta = D^{n-1}$.

Solution by W. E. HEAL, Wheeling, Delaware County, Indiana; A. R. BULLIS, Ithaca, Tompkins County, New York; H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C.; and WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

If we multiply the determinants $\varDelta = \Sigma \pm S_{11}S_{22}\ldots\ldots S_{nn}$, $D = \Sigma \pm a_{11}a_{22}\ldots\ldots a_{nn}$ together we get the determinant $\Sigma \pm c_{11}c_{22}\ldots\ldots c_{nn}$ in which the constituents are formed by the rule

$$c_{ik} = S_{i1}a_{k1} + S_{i2}a_{k2} + \ldots\ldots + S_{in}a_{kn}.$$

Now if $i = k$, $c_{ik} = D$, and if i does not $= k$, $c_{ik} = 0$. Thus the determinant $\Sigma \pm c_{11}c_{22}\ldots\ldots c_{nn}$ reduces to its first element $c_{11}c_{22}\ldots\ldots c_{nn}$, that is to D^n. $\therefore\ \varDelta D = D^n$, and $\varDelta = D^{n-1}$.

147.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Pa.

A right cone, whose equation is $x^2 + y^2 = e^2z^2 \ldots\ldots (1)$, and a paraboloid of revolution, whose equation is $y^2 + z^2 = px \ldots\ldots (2)$, have their vertices coincident, the axis of the cone being perpendicular to the axis of the paraboloid. Find the volume common to both by the formula $V = \iiint dxdydz$.

Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

The limits of integration for z are from $z = \dfrac{\sqrt{(x^2 + y^2)}}{e}$ to $z = \sqrt{(px - y^2)}$. Eliminating z from (1), (2), the limits of y are from $y=-\frac{\sqrt{(e^2px-x^2)}}{\sqrt{(1+e^2)}}$ to $y=+\frac{\sqrt{(e^2px-x^2)}}{\sqrt{(1+e^2)}}$. The greatest and least values of x are in the plane xz, hence putting $y=0$, the limits of x are from $x=0$ to $x=e^2p$.

$$\therefore\ V=\iiint dxdydz,\ =\iint\left[\sqrt{(px-x^2)}-\frac{\sqrt{(x^2+y^2)}}{e}\right]dxdy,$$

$$=\int\left[\tfrac{1}{2}y\sqrt{(px-x^2)}+\tfrac{1}{2}px\sin^{-1}\left(\frac{y}{\sqrt{(px)}}\right)-\frac{y\sqrt{(x^2+y^2)}}{2e}-\frac{x^2}{2e}\log[y+\sqrt{(x^2+y^2)}]\right]dx,$$

$$=\int\left[px\sin^{-1}\left(\frac{\sqrt{(e^2p-x)}}{\sqrt{[p(1+e^2)]}}\right)-\frac{x^2}{e}\log\left(\frac{e\sqrt{(p+x)}+\sqrt{(e^2p-x)}}{\sqrt{[x(1+e^2)]}}\right)\right]dx,$$

$$=\left[\tfrac{1}{2}px^2\sin^{-1}\left(\frac{\sqrt{(e^2p-x)}}{\sqrt{[p(1+e^2)]}}\right)-\frac{x^3}{3e}\log\left(\frac{e\sqrt{(p+x)}+\sqrt{(e^2p-x)}}{\sqrt{[x(1+e^2)]}}\right)\right.$$
$$\left.-\tfrac{1}{48}p^3(3e^4-2e^2+3)\tan^{-1}\left(\frac{\sqrt{(e^2p-x)}}{\sqrt{(p+x)}}\right)+\tfrac{1}{16}p[\tfrac{3}{4}p(1-e^2)-\tfrac{1}{2}x]\right]_0^{e^2p},$$

$$=\tfrac{1}{48}p^3[(3e^4-2e^2+3)\tan^{-1}e-3e(1-e^2)].$$

Excellent solutions were also given by Messrs. *Bowser*, *Heaton*, *Kummell*, *Ludwig* and *Seitz*.

SOLUTION OF "UNSOLVED" PROBLEM 40.

By E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Adair County, Missouri.

40. The first of two casks contains a gallons of wine, and the second b gallons of water. From the first is poured into the second as many gallons as it already contains, and then as much is poured from the second into the first as was left in the first. How much wine remains in the second cask after n such operations?

Let $u_n=$ the number of gallons of the mixture in the second cask after n operations, $v_n=$ the strength of the contents of the second cask, and $w_n=$ strength of the contents of the first cask.

Then $a+b-u_n=$ the number of gallons of the mixture in the first cask, and $u_nv_n=$ the number of gallons of wine in the second cask.

After the $(n+1)$th operation the number of gallons of the mixture in the second cask is

$$u_{n+1}=2u_n-(a+b-2u_n),\quad\text{whence}\quad u_{n+1}-4u_n=-(a+b)\ \ldots\ldots\ (1),$$

an equation in Finite Differences, whose solution gives

$$u_n=2^{2n}C+\tfrac{1}{3}(a+b).\quad\text{When } n=1,\ u_1=4C+\tfrac{1}{3}(a+b)=3b-a;\ \therefore\ C=\tfrac{2}{3}b-\tfrac{1}{3}a,$$

and

$$u_n=\tfrac{1}{3}b(2^{2n+1}+1)-\tfrac{1}{3}a(2^{2n}-1)\ \ldots\ldots\ (2).$$

The $(n+1)$th and $(n+2)$th operations give the following relations:

$$v_{n+1}=\tfrac{1}{2}(v_n+w_n)\ \ldots\ (3),\quad w_{n+1}=\tfrac{1}{2}(v_{n+1}+w_n)\ \ldots\ (4),\quad v_{n+2}=\tfrac{1}{2}(v_{n+1}+w_{n+1})\ \ldots\ (5).$$

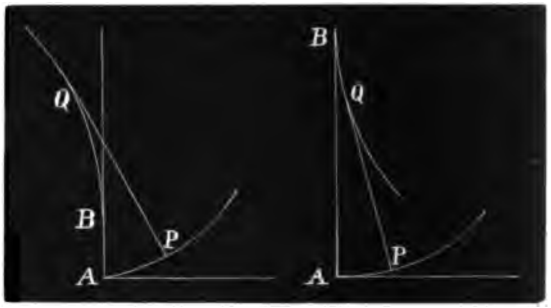
From (3), (4) and (5), by eliminating w_n and w_{n+1}, we find $4v_{n+2}-5v_{n+1}+v_n=0$ ......(6).

Solving (6) we find $v_n=C_1+C_2(\tfrac{1}{2})^{2n}$ ......(7). When $n=1$, $v_1=C_1+\tfrac{1}{4}C_2=\tfrac{1}{2}$ ......(8), and when $n=2$, $v_2=C_1+\tfrac{1}{16}C_2=\tfrac{5}{8}$ ......(9). From (8) and (9) we find $C_1=\tfrac{2}{3}$, and $C_2=-\tfrac{2}{3}$;

$$\therefore\ v_n=\tfrac{2}{3}\left(1-\frac{1}{2^{2n}}\right),\ \text{and}\ u_nv_n=\tfrac{2}{9}\left(1-\frac{1}{2^{2n}}\right)\left[b(2^{2n+1}+1)-a(2^{2n}-1)\right].$$

ON EVOLUTES AND INVOLUTES.

By ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pennsylvania.



IF the intrinsic equation to a curve be known, that of the evolute and involute can be found.

If ρ be the radius of curvature of the curve at the point determined by s and φ, we have (*Todhunter's Diff. Cal.*, Art. 324) $\rho = \frac{ds}{d\varphi}$.

"Let AP be a curve, BQ the evolute; let s be the length of an arc of AP measured from some fixed point up to P; s' the length of an arc of BQ measured from some fixed point up to Q. It is evident that φ is the same both for s and s' if in BQ we measure φ from BA, which is perpendicular to the straight line from which φ is measured in AP.

In the left-hand figure, $s' = \rho - C = \frac{ds}{d\varphi} - C$. In the right-hand figure, $s' = C - \rho = C - \frac{ds}{d\varphi}$.

Thus if s be known in terms of φ, we can find s' in terms of φ. The constant C is equal to the value of ρ at the point corresponding to that for which $s' = 0$". *Todhunter's Int. Cal.*, Art. 114.

Examples.—1. Find the equation to the evolute of the tractrix.

The intrinsic equation to the tractrix (see MATH. VISITOR, 2d edition of No. 1, p. 11) is $s = a\log\sec\varphi$; therefore $\frac{ds}{d\varphi} = a\tan\varphi$, and $s' = a\tan\varphi - C$. When $s' = 0$, $\varphi = 0$; $\therefore$ $C = 0$, and $s' = a\tan\varphi$ is the intrinsic equation to the evolute of the tractrix, which is, therefore, the catenary.

2. Required the equation to the evolute of the catenary.

We have $\frac{ds}{d\varphi} = a\sec^2\varphi$, therefore $s' = a\sec^2\varphi - C$. When $s' = 0$, $\varphi = 0$, and $C = a$; therefore $s' = a(\sec^2\varphi - 1)$, $= a\tan^2\varphi$, is the intrinsic equation to the evolute of the catenary.

$$x = \int\frac{2a\sin^2\varphi d\varphi}{\cos^3\varphi} = \frac{a\sin\varphi}{1-\sin^2\varphi} - \tfrac{1}{2}a\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) + C', \quad y = \int\frac{2a\sin\varphi d\varphi}{\cos^2\varphi} = \frac{2a}{\cos\varphi} + C''.$$

When $\varphi = 0$, $x = 0$, $y = 0$, $\therefore$ $C' = 0$ and $C'' = -2a$; hence

$$x = \frac{a\sin\varphi}{1-\sin^2\varphi} - \tfrac{1}{2}a\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right), \quad y = \frac{2a}{\cos\varphi} - 2a. \quad \therefore \cos\varphi = \frac{2a}{y+2a}, \quad \sin\varphi = \frac{\sqrt{(y^2+4ay)}}{y+2a} \text{ and}$$

$$x = \frac{(y+2a)\sqrt{(y^2+4ay)}}{4a} - \tfrac{1}{2}a\log\left(\frac{y+2a+\sqrt{(y^2+4ay)}}{y+2a-\sqrt{(y^2+4ay)}}\right),$$

the rectangular equation to the evolute of the catenary.

If $y = 2a$ when $x = 0$, the equation is

$$x = \frac{y\sqrt{(y^2-4a^2)}}{4a} - \tfrac{1}{2}a\log\left(\frac{y^2-2a^2+y\sqrt{(y^2-4a^2)}}{4a^2}\right).$$

To find the intrinsic equation to the *involute* of a curve we have $\frac{ds}{d\varphi} = C \pm s'$; $\therefore$ $s = \int(C \pm s')d\varphi$.

Examples.—1. Determine the equation to the involute of the circle.

The intrinsic equation to the circle is $s' = a\varphi$; $\therefore$ $s = \int(C \pm a\varphi)d\varphi = C\varphi \pm \tfrac{1}{2}a\varphi^2 + C'$, $= \tfrac{1}{2}a\varphi^2$ if we suppose s to begin where the involute meets the circle and $\varphi = 0$ at that point.

Since the axis of x is a tangent to the involute where it begins, we have

$$x = r\cos\theta = \int\cos\varphi ds = \int a\varphi\cos\varphi d\varphi = a(\varphi\sin\varphi + \cos\varphi) + C,$$

$$y = r\sin\theta = \int\sin\varphi ds = \int a\varphi\sin\varphi d\varphi = a(-\varphi\cos\varphi + \sin\varphi) + C'.$$

When $\varphi = 0,\ x = a,\ y = 0$; therefore $C = 0$ and $C' = 0$.

$$\therefore\quad \varphi\sin\varphi + \cos\varphi = \frac{x}{a} \ldots\ldots\ldots\ldots(1), \qquad -\varphi\cos\varphi + \sin\varphi = \frac{y}{a} \ldots\ldots\ldots\ldots(2).$$

Adding squares of (1) and (2),

$$\varphi^2 + 1 = \frac{x^2+y^2}{a^2} = \frac{r^2}{a^2}, \quad \text{and} \quad \varphi = \frac{\sqrt{(r^2-a^2)}}{a} \ldots\ldots\ldots\ldots(3).$$

Multiplying (1) by $\sin\varphi$ and (2) by $\cos\varphi$, and taking the difference of the results,

$$\varphi = \frac{r\cos\theta\sin\varphi - r\sin\theta\cos\varphi}{a} = \frac{r\sin(\varphi-\theta)}{a};$$

$$\therefore\quad \sin(\varphi-\theta) = \frac{\sqrt{(r^2-a^2)}}{r}, \quad \text{whence} \quad \varphi - \theta = \sin^{-1}\left(\frac{\sqrt{(r^2-a^2)}}{r}\right) = \cos^{-1}\left(\frac{a}{r}\right),$$

and therefore by (3)

$$\theta = \frac{\sqrt{(r^2-a^2)}}{a} - \cos^{-1}\left(\frac{a}{r}\right),$$

the polar equation to the involute of the circle.

2. **Required the equation to the *involute* of the involute of the circle.**

The intrinsic equation of the involute of the circle being $s' = \frac{1}{2}a\varphi^2$, that of the *involute* of the involute is $s = \int\frac{1}{2}a\varphi^2 d\varphi = \frac{1}{6}a\varphi^3$, supposing both involutes to begin when $\varphi = 0$; and, *generally*, the intrinsic equation to the nth involute is $s = \dfrac{1}{1.2.3\ldots(n+n)}a\varphi^{n+1}$.

The axis of y is parallel to a tangent to the involute where it begins,

$$\therefore\ x = r\cos\theta = \int\sin\varphi\, ds = \tfrac{1}{2}a\int\varphi^2\sin\varphi\, d\varphi = \tfrac{1}{2}a(-\varphi^2\cos\varphi + 2\varphi\sin\varphi + 2\cos\varphi)\ldots(1),$$

$$y = r\sin\theta = \int -\cos\varphi\, ds = \tfrac{1}{2}a\int -\varphi^2\cos\varphi\, d\varphi = \tfrac{1}{2}a(-\varphi^2\sin\varphi - 2\varphi\cos\varphi + 2\sin\varphi)\ldots(2).$$

Adding squares of (1) and (2),

$$a^2(\varphi^4 + 4) = 4r^2, \quad \text{and} \quad \varphi = \left(\frac{4(r^2-a^2)}{a^2}\right)^{\frac{1}{4}} \ldots\ldots\ldots\ldots(3).$$

Multiplying (1) by $\sin\varphi$ and (2) by $\cos\varphi$, and taking the difference of the results,

$$r\sin\varphi\cos\theta - r\cos\varphi\sin\theta = a\varphi \ldots\ldots\ldots\ldots(4);$$

whence

$$r\sin(\varphi-\theta) = a\varphi, \quad \text{and} \quad \varphi - \theta = \sin^{-1}\left(\frac{a\varphi}{r}\right) \ldots\ldots\ldots\ldots(5).$$

Substituting value of φ from (3),

$$\theta = \left(\frac{4(r^2-a^2)}{a^2}\right)^{\frac{1}{4}} - \sin^{-1}\left(\frac{4a^2(r^2-a^2)}{r^4}\right)^{\frac{1}{4}}.$$

3. **Required the equation to the involute of the parabola.**

The rectangular equation to the parabola being $y^2 = 4px$, its intrinsic equation is

$$s' = \frac{p\sin\varphi}{1-\sin^2\varphi} + \tfrac{1}{2}p\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \ldots\ldots\ldots\ldots(1)$$

and that of the involute is

$$s = \int\frac{p\sin\varphi\, d\varphi}{1-\sin^2\varphi} + \int\tfrac{1}{2}p\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)d\varphi \ldots\ldots\ldots\ldots(2).$$

$$x = \int\cos\varphi\, ds = \int\frac{p\cos\varphi\sin\varphi\, d\varphi}{1-\sin^2\varphi} + \int\tfrac{1}{2}p\cos\varphi\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)d\varphi = \tfrac{1}{2}p\sin\varphi\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) \ldots\ldots(3),$$

$$y = \int\sin\varphi\, ds = \int\frac{p\sin^2\varphi\, d\varphi}{1-\sin^2\varphi} + \int\tfrac{1}{2}p\sin\varphi\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)d\varphi = \frac{p\sin\varphi}{\sqrt{(1-\sin^2\varphi)}} - \tfrac{1}{2}p\cos\varphi\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right),$$

supposing $x = 0$ and $y = 0$ when $\varphi = 0$.

From the preceding equations we get

$$\tfrac{1}{2}p\log\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right) = \frac{x}{\sin\varphi} = \frac{p\sin\varphi}{\cos^2\varphi} - \frac{y}{\cos\varphi}; \quad \text{whence} \quad (x+p)\sin^2\varphi - x = y\sin\varphi\cos\varphi.$$

Squaring and solving we find

$$\sin\varphi = \frac{(2x^2+2px+y^2) \pm \sqrt{[(2x^2+2px+y^2)^2 - 4x^2(x^2+2px+y^2+p^2)]}}{2(x+p)^2+2y^2}.$$

Substituting in (3),

$$\frac{2x(x+p)^2+2xy^2}{(2x^2+2px+y^2) \pm \sqrt{[(2x^2+2px+y^2)^2-4x^2(x^2+2px+y^2+p^2)]}}$$
$$= \tfrac{1}{2}p\log\left(\frac{(4x^2+6px+3y^2+2p^2) \pm \sqrt{[(2x^2+2px+y^2)^2-4x^2(x^2+2px+y^2+p^2)]}}{(y^2+2px+2p^2) \mp \sqrt{[(2x^2+2px+y^2)^2-4x^2(x^2+2px+y^2+p^2)]}}\right).$$

4. **Required the equation to the involute of the ellipse.**

The rectangular equation to the ellipse, the origin being at the vertex, is

$$ay = b\sqrt{(2ax-x^2)};\ \therefore\ \frac{dy}{dx} = \frac{b(a-x)}{a\sqrt{(2ax-x^2)}} = \cot\varphi \quad\ldots\ldots\ldots\ldots(1).$$

From (1), $x = a \pm \dfrac{a^2}{\sqrt{(a^2+b^2\tan^2\varphi)}} = a - \dfrac{a^2\cos\varphi}{\sqrt{[a^2-(a^2-b^2)\sin^2\varphi]}}$, taking the lower sign;

$$\therefore \frac{dx}{d\varphi} = \frac{a^2b^2\sin\varphi}{[a^2-(a^2-b^2)\sin^2\varphi]^{\frac{3}{2}}}.\quad \text{But } \frac{ds'}{dx} = \frac{1}{\sin\varphi},\ \therefore \frac{ds'}{d\varphi} = \frac{a^2b^2}{[a^2-(a^2-b^2)\sin^2\varphi]^{\frac{3}{2}}} = \frac{b^2}{a(1-e^2\sin^2\varphi)^{\frac{3}{2}}}.$$

Integrating,

$$s' = a\mathrm{E}(e,\varphi) - \frac{e^2a\sin\varphi\cos\varphi}{\sqrt{(1-e^2\sin^2\varphi)}} \quad\ldots\ldots\ldots\ldots(2),$$

the intrinsic equation of the ellipse.

For the involute we have $ds = s'd\varphi = a\mathrm{E}(e,\varphi)d\varphi - \dfrac{e^2a\sin\varphi\cos\varphi\,d\varphi}{\sqrt{(1-e^2\sin^2\varphi)}} \quad\ldots\ldots\ldots\ldots(3)$,

but $\mathrm{E}(e,\varphi)d\varphi$ does not appear to be integrable. We can, however, find the polar and rectangular equations.

Since the axis of x is a tangent to the involute where it begins,

$$x = r\cos\theta = \int\cos\varphi\,ds = a\int\left(\mathrm{E}(e,\varphi) - \frac{e^2\sin\varphi\cos\varphi}{\sqrt{(1-e^2\sin^2\varphi)}}\right)\cos\varphi\,d\varphi \quad\ldots\ldots\ldots\ldots(4),$$

$$y = r\sin\theta = \int\sin\varphi\,ds = a\int\left(\mathrm{E}(e,\varphi) - \frac{e^2\sin\varphi\cos\varphi}{\sqrt{(1-e^2\sin^2\varphi)}}\right)\sin\varphi\,d\varphi \quad\ldots\ldots\ldots\ldots(5).$$

Integrating, and observing that when $\varphi = 0$, $x = a$ and $y = 0$,

$$x = r\cos\theta = a\sin\varphi\mathrm{E}(e,\varphi) + a\cos\varphi\sqrt{(1-e^2\sin^2\varphi)} \quad\ldots\ldots\ldots\ldots(6),$$
$$y = r\sin\theta = -a\cos\varphi\mathrm{E}(e,\varphi) + a\sin\varphi\sqrt{(1-e^2\sin^2\varphi)} \quad\ldots\ldots\ldots\ldots(7).$$

Multiplying (6) by $\sin\varphi$ and (7) by $\cos\varphi$ and taking the difference of the results,

$$x\sin\varphi - y\cos\varphi = r\sin\varphi\cos\theta - r\sin\theta\cos\varphi = r\sin(\varphi-\theta) = a\mathrm{E}(e,\varphi) \quad\ldots\ldots\ldots\ldots(8).$$

Multiplying (6) by $\cos\varphi$ and (7) by $\sin\varphi$ and taking the sum of the products,

$$x\cos\varphi + y\sin\varphi = r\cos(\varphi-\theta) = a\sqrt{(1-e^2\sin^2\varphi)} \quad\ldots\ldots\ldots\ldots(9).$$

From (8),

$$\theta = \varphi - \sin^{-1}\left(\frac{a\mathrm{E}(e,\varphi)}{r}\right) \quad\ldots\ldots\ldots\ldots(10).$$

Squaring (9) we get

$$2r^2\sin\theta\cos\theta\sqrt{(\sin^2\varphi-\sin^4\varphi)} = a^2 - r^2\cos^2\theta + (r^2\cos2\theta - a^2e^2)\sin^2\varphi \quad\ldots\ldots\ldots\ldots(11).$$

Squaring again and solving the resulting equation, we find

$$\sin\varphi = \left(\frac{a^2(r^2+a^2e^2) + r^2(r^2-2a^2-a^2e^2)\cos^2\theta + ar^2\sin2\theta\sqrt{[r^2(1-e^2\cos^2\theta)-a^2(1-e^2)]}}{r^4+a^4e^4-2a^2r^2e^2\cos2\theta}\right)^{\frac{1}{2}},$$

$$\cos\varphi = \pm\left(\frac{r^4-a^4e^2(1-e^2)-a^2r^2(1-2e^2)-r^2(r^2-2a^2+3a^2e^2)\cos^2\theta - ar^2\sin2\theta\sqrt{[r^2(1-e^2\cos^2\theta)-a^2(1-e^2)]}}{r^4+a^4e^4-2a^2r^2e^2\cos2\theta}\right)^{\frac{1}{2}}.$$

Substituting in (8),

$$\cos\theta\left(\frac{a^2(r^2+a^2e^2)+r^2(r^2-2a^2-a^2e^2)\cos^2\theta+ar^2\sin2\theta\sqrt{[r^2(1-e^2\cos^2\theta)-a^2(1-e^2)]}}{r^4+a^4e^4-2a^2r^2e^2\cos2\theta}\right)^{\frac{1}{2}}$$
$$\pm\sin\theta\left(\frac{r^4-a^4e^2(1-e^2)-a^2r^2(1-2e^2)-r^2(r^2-2a^2+3a^2e^2)\cos^2\theta-ar^2\sin2\theta\sqrt{[r^2(1-e^2\cos^2\theta)-a^2(1-e^2)]}}{r^4+a^4e^4-2a^2r^2e^2\cos2\theta}\right)^{\frac{1}{2}}$$
$$= \frac{a}{r}\mathbf{E}\left[e,\ \sin^{-1}\left(\frac{a^2(r^2+a^2e^2)+r^2(r^2-2a^2-a^2e^2)\cos^2\theta+ar^2\sin2\theta\sqrt{[r^2(1-e^2\cos^2\theta)-a^2(1-e^2)]}}{r^4+a^4e^4-2a^2r^2e^2\cos2\theta}\right)^{\frac{1}{2}}\right].$$

I am indebted to Professor E. B. SEITZ for simplifying the value of $\sin\varphi$.

COMPUTATION OF STURM'S FUNCTIONS.

By W. E. HEAL, Marion, Grant County, Indiana.

THE following process (which so far as I know has never been published) seems to simplify, to some extent, the calculation of Sturm's functions.

Let $f_{m-2}(x) = a_1x^n + a_2x^{n-1} + a_3x^{n-2} + a_4x^{n-3} + \ldots\ldots + a_{n+1}$ ....(1),

$f_{m-1}(x) = b_1x^{n-1} + b_2x^{n-2} + b_3x^{n-3} + b_4x^{n-4} + \ldots\ldots + b_n$ ....(2),

$f_m\ (x) = c_1x^{n-2} + c_2x^{n-3} + c_3x^{n-4} + c_4x^{n-5} + \ldots\ldots + c_{n-1}$ ....(3),

be the $(m-2)$th, $(m-1)$th and mth Sturmian functions.

Multiply (2) by an arbitrary function, $px+q$, of the first degree in x, and we get

$$(px+q)f_{m-1}(x) = pb_1x^n + (pb_2+qb_1)x^{n-1} + (pb_3+qb_2)x^{n-2} + (pb_4+qb_3)x^{n-3} + \ldots\ldots + qb_n \ldots (4).$$

If we subtract (1) from (4) and determine p, q so that the two highest terms shall vanish we get (3).

$$\therefore\ f_m(x) = (qb_2+pb_3-a_3)x^{n-2} + (qb_3+pb_4-a_4)x^{n-3} + \ldots\ldots + (qb_n-a_{n+1}) \ldots\ldots (5),$$

in which $q = \dfrac{a_2b_1-a_1b_2}{b_1^2}$, $p = \dfrac{a_1}{b_1}$.

Multiply (5) by b_1^2, and we may take

$$f_m(x) = (rb_2+sb_3-b_1^2a_3)x^{n-2} + (rb_3+sb_4-b_1^2a_4)x^{n-3} + \ldots\ldots + (rb_n-b_1^2a_{n+1}) \ldots\ldots (6),$$

in which $r = a_2b_1 - a_1b_2$, $s = a_1b_1$.

Since a positive factor may be suppressed in any of Sturm's functions the last function may always be taken as one of the numbers $+1$ or -1.

The computation by (6) may be arranged as follows:

Given $F(x) = x^4 - 6x^3 + 5x^2 + 14x - 4 = 0$, to find Sturm's functions.

$F'(x) = 2x^3 - 9x^2 + 5x + 7$. $r_1 = (-6)(2) - (1)(-9) = -3$, $s_1 = 1\times2 = 2$.

$f_1(x) = 17x^2 - 57x - 5$. $r_2 = (-9)(17) - (2)(-57) = -39$, $s_2 = 2\times17 = 34$.

$f_2(x) = 152x - 457$. $r_3 = (-57)(152) - (17)(-457) = -895$.

$f_3(x) = 1$.

It is unnecessary to find s in computing the last function.

Given $F(x) = x^4 - 8x^3 + 14x^2 + 4x - 8 = 0$, to find Sturm's functions.

$F'(x) = x^3 - 6x^2 + 7x + 1$. $r_1 = (-8)(1) - (1)(-6) = -2$, $s_1 = 1\times1 = 1$.

$f_1(x) = 5x^2 - 17x + 6$. $r_2 = (-6)(5) - (1)(-17) = -13$, $s_2 = 1\times5 = 5$.

$f_2(x) = 76x - 103$. $r_3 = (-17)(76) - (5)(-103) = -777$.

$f_3(x) = 1$.

EXPANSION OF POLYNOMIALS.

By E. P. THOMPSON, M. E., Elizabeth, Union County, New Jersey.

THE present article is intended, not to set forth any new mathematical principles, but to establish rules whereby polynomials may be easily and rapidly expanded to any power.

The analysis of the method of obtaining these rules may be briefly stated, thus;—by expanding any polynomial to any power by means of the Binomial Theorem, the terms may be arranged in such an order, as to constitute a formula. From an inspection of this formula a few simple rules are deduced.

According to the Binomial Theorem, the expansion of $(\ldots . c+d+e+x)^n$ may be put in the form indicated on the left of the brackets and when expanded more completely, the formula becomes that which is shown on the right of the brackets. The latter formula is the one from which the rules are obtained.

$[\ldots c+(d+e+x)]^n =$

$$c^0(d+e+x)^n = \left\{ \begin{array}{rr} c^0d^0e^0x^n \; + & n\,c^0d^0e^1x^{n-1} \\ n\,c^0d^1e^0x^{n-1} + & n(n-1)\,c^0d^1e^1x^{n-2} \\ \frac{n(n-1)}{1.2}c^0d^2e^0x^{n-2} + & \frac{n(n-1)(n-2)}{1.2}c^0d^2e^1x^{n-3} \\ \frac{n(n-1)(n-2)}{1.2.3}c^0d^3e^0x^{n-3} + & \frac{n(n-1)\ldots(n-3)}{1.2.3}c^0d^3e^1x^{n-4} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ c^0d^ne^0x^0 & \end{array}\right.$$

$$+ \frac{n(n-1)}{1.2}c^0d^0e^2x^{n-2} + \frac{n(n-1)(n-2)}{1.2.3}c^0d^0e^3x^{n-3} + \ldots\ldots + c^0d^0e^nx^0$$

$$+ \frac{n(n-1)(n-2)}{1.2}c^0d^1e^2x^{n-3} + \frac{n(n-1)\ldots(n-3)}{1.2.3}c^0d^1e^3x^{n-4} + \ldots\ldots + 0$$

$$+ \frac{n(n-1)\ldots(n-3)}{1.2.1.2}c^0d^2e^2x^{n-4} + \frac{n(n-1)\ldots(n-4)}{1.2.3.1.2}c^0d^2e^3x^{n-5} + \ldots\ldots + 0$$

$$+ \frac{n(n-1)\ldots(n-4)}{1.2.1.2.3}c^0d^3e^2x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.3.1.2.3}c^0d^3e^3x^{n-6} + \ldots\ldots + 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+$$

$$n\,c^1(d+e+x)^{n-1} = \left\{ \begin{array}{rr} n\,c^1d^0e^0x^{n-1} + & n(n-1)\,c^1d^0e^1x^{n-2} \\ n(n-1)\,c^1d^1e^0x^{n-2} + & n(n-1)(n-2)\,c^1d^1e^1x^{n-3} \\ \frac{n(n-1)(n-2)}{1.2}c^1d^2e^0x^{n-3} + & \frac{n(n-1)\ldots(n-3)}{1.2}c^1d^2e^1x^{n-4} \\ \frac{n(n-1)\ldots(n-3)}{1.2.3}c^1d^3e^0x^{n-4} + & \frac{n(n-1)\ldots(n-4)}{1.2.3}c^1d^3e^1x^{n-5} \\ \ldots\ldots\ldots & \ldots\ldots\ldots \\ n\,c^1d^{n-1}e^0x^0 & \end{array}\right.$$

$$+ \frac{n(n-1)(n-2)}{1.2}c^1d^0e^2x^{n-3} + \frac{n(n-1)\ldots(n-3)}{1.2.3}c^1d^0e^3x^{n-4} + \ldots\ldots n\,c^1d^0e^{n-1}x^0$$

$$+ \frac{n(n-1)\ldots(n-3)}{1.2}c^1d^1e^2x^{n-4} + \frac{n(n-1)\ldots(n-4)}{1.2.3}c^1d^1e^3x^{n-5} + \ldots\ldots + 0$$

$$+ \frac{n(n-1)\ldots(n-4)}{1.2.1.2}c^1d^2e^2x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.3.1.2}c^1d^2e^3x^{n-6} + \ldots\ldots + 0$$

$$+ \frac{n(n-1)\ldots(n-5)}{1.2.1.2.3}c^1d^3e^2x^{n-6} + \frac{n(n-1)\ldots(n-6)}{1.2.3.1.2.3}c^1d^3e^3x^{n-7} + \ldots\ldots + 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+$$

$$\frac{n(n-1)}{1.2}c^2(d+e+x)^{n-2} = \left\{\begin{array}{l}
\frac{n(n-1)}{1.2}c^2d^0e^0x^{n-2} + \frac{n(n-1)(n-2)}{1.2}c^2d^0e^1x^{n-3} \\
\frac{n(n-1)(n-2)}{1.2}c^2d^1e^0x^{n-3} + \frac{n(n-1)\ldots(n-3)}{1.2}c^2d^1e^1x^{n-4} \\
\frac{n(n-1)\ldots(n-3)}{1.2.1.2}c^2d^2e^0x^{n-4} + \frac{n(n-1)\ldots(n-4)}{1.2.1.2}c^2d^2e^1x^{n-5} \\
\frac{n(n-1)\ldots(n-4)}{1.2.1.2.3}c^2d^3e^0x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.1.2.3}c^2d^3e^1x^{n-6} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{n(n-1)}{1.2}c^2d^{n-2}e^0x^0
\end{array}\right.$$

$$+\frac{n(n-1)\ldots(n-3)}{1.2.1.2}c^2d^0e^2x^{n-4} + \frac{n(n-1)\ldots(n-4)}{1.2.3.1.2}c^2d^0e^3x^{n-5} + \ldots + \frac{n(n-1)}{1.2}c^2d^0e^{n-2}x^0$$

$$+\frac{n(n-1)\ldots(n-4)}{1.2.1.2}c^2d^1e^2x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.3.1.2}c^2d^1e^3x^{n-6} + \ldots\ldots\ldots + 0$$

$$+\frac{n(n-1)\ldots(n-5)}{1.2.1.2.1.2}c^2d^2e^2x^{n-6} + \frac{n(n-1)\ldots(n-6)}{1.2.3.1.2.1.2}c^2d^2e^3x^{n-7} + \ldots\ldots\ldots + 0$$

$$+\frac{n(n-1)\ldots(n-6)}{1.2.1.2.1.2.3}c^2d^3e^2x^{n-7} + \frac{n(n-1)\ldots(n-7)}{1.2.3.1.2.1.2.3}c^2d^3e^3x^{n-8} + \ldots\ldots\ldots + 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+$$

$$\frac{n(n-1)(n-2)}{1.2.3}c^3(d+e+x)^{n-3} = \left\{\begin{array}{l}
\frac{n(n-1)(n-2)}{1.2.3}c^3d^0e^0x^{n-3} + \frac{n(n-1)\ldots(n-3)}{1.2.3}c^3d^0e^1x^{n-4} \\
\frac{n(n-1)\ldots(n-3)}{1.2.3}c^3d^1e^0x^{n-4} + \frac{n(n-1)\ldots(n-4)}{1.2.3}c^3d^1e^1x^{n-5} \\
\frac{n(n-1)\ldots(n-4)}{1.2.3.1.2}c^3d^2e^0x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.3.1.2}c^3d^2e^1x^{n-6} \\
\frac{n(n-1)\ldots(n-5)}{1.2.3.1.2.3}c^3d^3e^0x^{n-6} + \frac{n(n-1)\ldots(n-6)}{1.2.3.1.2.3}c^3d^3e^1x^{n-7} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\frac{n(n-1)(n-2)}{1.2.3}c^3d^{n-3}e^0x^0
\end{array}\right.$$

$$+\frac{n(n-1)\ldots(n-4)}{1.2.3.1.2}c^3d^0e^2x^{n-5} + \frac{n(n-1)\ldots(n-5)}{1.2.3.1.2.3}c^3d^0e^3x^{n-6} + \ldots + \frac{n(n-1)(n-2)}{1.2.3}c^3d^0e^{n-3}x^0$$

$$+\frac{n(n-1)\ldots(n-5)}{1.2.3.1.2}c^3d^1e^2x^{n-6} + \frac{n(n-1)\ldots(n-6)}{1.2.3.1.2.3}c^3d^1e^3x^{n-7} + \ldots\ldots\ldots + 0$$

$$+\frac{n(n-1)\ldots(n-6)}{1.2.3.1.2.1.2}c^3d^2e^2x^{n-7} + \frac{n(n-1)\ldots(n-7)}{1.2.3.1.2.3.1.2}c^3d^2e^3x^{n-8} + \ldots\ldots\ldots + 0$$

$$+\frac{n(n-1)\ldots(n-7)}{1.2.3.1.2.1.2.3}c^3d^3e^2x^{n-8} + \frac{n(n-1)\ldots(n-8)}{1.2.3.1.2.3.1.2.3}c^3d^3e^3x^{n-9} + \ldots\ldots\ldots + 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$c^n$$

Example.—$(c+d+e+x)^3 = c^0d^0e^0x^3 + 3c^0d^0e^1x^2 + 3c^0d^0e^2x^1 + c^0d^0e^3x^0$

$3c^0d^1e^0x^2 + 6c^0d^1e^1x^1 + 3c^0d^1e^2x^0$

$3c^0d^2e^0x^1 + 3c^0d^2e^1x^0$

$c^0d^3e^0x^0$

$3c^1d^0e^0x^2 + 6c^1d^0e^1x^1 + 3c^1d^0e^2x^0$

$6c^1d^1e^0x^1 + 6c^1d^1e^1x^0$

$3c^1d^2e^0x^0$

$3c^2d^0e^0x^1 + 3c^2d^0e^1x^0$

$3c^2d^1e^0x^0$

$c^3d^0e^0x^0$

As might be expected, this formula is long and appears, at first glance, to be entirely without regularity; but notice nine laws.

(1.) In each bracket, the exponent of x diminishes by 1 from left to right in each horizontal line and also from top to bottom in each vertical column, while the exponent of e increases by 1 in each horizontal line from left to right and the exponent of d increases by 1 in each column from top to bottom. Upon the exponents of these three letters (d, e, x) depend the coefficients; for—

(2.) If the coefficient of any term is multiplied by the exponent of x and the product divided by the exponent of e increased by 1, then the quotient is equal to the coefficient of the succeeding term in the same *horizontal* line; while, by dividing the product by the exponent of d increased by 1, the coefficient of the succeeding term in the same *vertical* column is the result. The coefficients of the first term in each bracket are the same and in the same order as those of a binomial raised to the nth power, *i. e.*,

$1,\ n,\ \frac{n(n-1)}{1.2},\ \frac{n(n-1)(n-2)}{1.2.3}$, etc.

(3.) The exponents of c are 0 in the first bracket, 1 in the second bracket, 2 in the third, ... n in the $(n+1)$th.

(4.) The exponents in each term sum up to n.

(5.) The first horizontal line in the first bracket is equal to the expansion of the *binomial* $(e+x)$ to the nth power; all the terms in the first bracket are equal to the expansion of the *trinomial* $(d+e+x)$ to the nth power; while all the terms taken together represent the expansion of the *tetranomial* $(c+d+e+x)$ to the nth power.

(6.) In any bracket, the coefficients of the first vertical column are the same and in the same order as those of the first horizontal line. In performing any example, the terms in each bracket will arrange themselves in the form of concentric triangles. The three sides of each concentric triangle are made up of terms containing the same coefficients and in the same order. Moreover, the coefficients of the first column in the second bracket are the same and in the same order as the coefficients of the second column in the first bracket; in the same manner, those of the first column in the third bracket are the same and in the same order as those of the third column in the first bracket, and those of the first column in the fourth bracket are the same and in the same order as those of the fourth column in the first bracket, etc. This law will facilitate the performing of an example, in that, only a few coefficients need be found by the second law—the rest of them are found by copying. This law will be more easily comprehended by referring to the example as well as to the formula.

(7.) Let e = exponent of x, $= (n-e'-e''-e''')$, e' = exponent of e, e'' = exponent of d, e''' = exponent of c. Then, according to the formula, the general term

$$T = \frac{n(n-1)(n-2)\ldots\ldots(n-e'-e''-e'''+1)}{1.2.3\ldots.e'.1.2.3\ldots.e''.1.2.3\ldots.e'''} c^{e'''}d^{e''}e^{e'}x^{n-e'-e''-e'''}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(e+1)}{1.2.3\ldots.e'.1.2.3\ldots.e''.1.2.3\ldots.e'''} c^{e'''}d^{e''}e^{e'}x^{e}.$$

Thus the coefficient of $b^2c^0d^0e^1x^2$, in the expansion of $(b+c+d+e+x)^5$, is equal to $\frac{5(4)(3)}{2.1.1} = 30$.

(8.) The number of terms in an expanded polynomial of t terms is equal to the number of combinations of $(n+t-1)$ quantities taken $(t-1)$ at a time. (See my problem, 299, in the *Analyst*, and its solutions, Vol. VII, No. 2, p. 98.)

Thus, the number of terms in $(a \pm b \pm c \pm d \pm e \pm x)^6$ is equal to $\frac{12.11.10.9.8.7}{1.2.3.4.5.6} = 924$.

(9.) The sum of the coefficients, considering them as all having the same sign, is always equal to t^n; for, if $x = e = d = c = 1$ then $\ldots.(c+d+e+x)^n = 4^n$.

The numerical values of the coefficients in the expanded polynomial are independent of the signs.

These last two laws, especially, should be applied to an example performed, in order to test the accuracy of the work.

In performing an example the zero powers of d, e and x should not be omitted, as they are necessary for obtaining some of the coefficients according to law (2.)

In order to make the manual labor rapid, observe the following outline: (Reference should be made to the given example.)

(A). Write the product (*bcdex*) repeatedly, in such order, as taken together, they form sets of concentric triangles as represented by the following dots:

```
......  .....  ....  ...  ..  .    .....  ....  ...  ..  .    ....  ...  ..  .    ...  ..  .    ..  .    .
.....   ....   ...   ..   .        ....   ...   ..   .        ...   ..   .        ..   .        .
....    ...    ..    .             ...    ..    .             ..    .             .
...     ..     .                   ..     .                   .
..      .                          .
.
```

(B). Write the exponents of each letter separately, thus; we give x the following exponents in the order represented:

```
5 4 3 2 1 0      4 3 2 1 0      3 2 1 0      2 1 0
4 3 2 1 0        3 2 1 0        2 1 0        1 0
3 2 1 0          2 1 0          1 0          0
2 1 0            1 0            0
1 0              0
0
```

&c. according to law (1) and then write the exponents of the other letters separately.

(C). Annex the coefficients according to laws (2) and (6).

(D). The signs of the terms of the expanded polynomial are determined by the signs of the powers and products of the positive and negative terms of the unexpanded polynomial, and may be found according to the two algebraic rules—

(1). The product of like signs gives plus, and of unlike signs, minus.

(2). If the quantity be negative, make the even powers positive and the odd powers negative.

In the expansion of $(a-b+c+d-e-f+g-x)^8$, it is evident that the term $-3380a^0b^1c^0d^3e^2f^1g^0x^1$ is negative.

By becoming familiar with the above laws and rules, the expansion of polynomials may be performed with wonderful rapidity.

THE DAY OF THE WEEK FOR A GIVEN DATE.

By Frank T. Freeland, B. S., Instructor in Mechanics, University of Pennsylvania, Philadelphia, Pa.

Since January 1st, 1877, is the second day of the week, it will be found that if "1" be added to a day of the month, the remainder obtained by dividing this sum by "7" will be the day of the week. "1" may be called the *ruling* number of January, 1877. In general, if $b=$ the ruling number of any month, $M=$ the day of the month, $n=$ the quotient obtained by dividing by "7", and $W=$ the day of the week; then

$$b+M=7n+W. \qquad \text{(I)}$$

Since February 1st is the fifth day of the week, the ruling number of February, 1877, is "4", which is three more than that of January. Since March 1st is the fifth day of the week, the ruling number of March, 1877, is "4", three more than that of January, &c., &c. Therefore, if a represents the ruling number of January of any year, which may be conceived to be the ruling number of the year, then

$b=a$ in January,	$a+1$ in May,	$a+5$ in September,	(II)
$a+3$ in February,	$a+4$ in June,	a in October,	
$a+3$ in March,	$a+6$ in July,	$a+3$ in November,	
$a+6$ in April,	$a+2$ in August,	$a+5$ in December.	

In leap years the ruling numbers of all the months except January and February will be one more..

According to the New-Style Calendar, every year has 365 days, every fourth year 366 days, every one-hundredth year 365 days and every four-hundredth year 366 days. Now 365 is of the form of $7n+1$. Hence after every completed common year a will be one more, and after every completed leap year two more. But $a=1$ for 1877 is the result of these additions to the ruling number of the year 1, A. D. If p represents this number, then

$$p+1877-1+\tfrac{1}{4}(1877-1)-\tfrac{1}{100}(1877-1)+\tfrac{1}{400}(1877-1)=7n+1,$$

which gives $p=1$.

According to the Old-Style Calendar, every year has 365 days and every fourth year 366 days. January 1st, 1877, Old Style, is seventh day. Hence "6" is the Old-Style ruling number of January, 1877. If $q=$ the Old-Style ruling number of 1 A. D., then

$$q+1877-1+\tfrac{1}{4}(1877-1)=7n+6,$$

which gives $q=6$.

Therefore if Y = any year and a = the ruling number of that year, then for New-Style dates

$$Y + \tfrac{1}{4}(Y-1) - \tfrac{1}{100}(Y-1) + \tfrac{1}{400}(Y-1) = 7n + a, \qquad \text{(III)}$$

and for Old-Style dates

$$Y + 5 + \tfrac{1}{4}(Y-1) = 7n + a, \qquad \text{(IV)}$$

because the New-Style ruling number of the year Y is the result of these additions to "1", the ruling number of the year 1, A. D., and the Old-Style ruling number of the year Y is the result of these additions to "6", the Old-Style ruling number of 1 A. D.

In finding the value of the fractions the decimals must be neglected.

Example.—Burr fought the duel with Hamilton July 11, 1804. What was the day of the week?

Substituting 1804 for Y in (III) we have

$$1804 + 450 - 18 + 4 = 2240 = 7n + a.$$

Dividing 2240 by 7 the remainder is 0, therefore $a = 0$.

Substituting in (II) for July we have $b = 0 + 6 + 1$ (for leap year) $= 7$. Substituting in (I) we have $7 + 11 = 18 = 7n + W$. Dividing 18 by 7 the remainder is 4, therefore $W = 4$ = Wednesday.

This result may be compared with pp. 347 and 357 of Parton's Burr, Vol. I.

The years in any century in which February has five Sundays are given by the formula

$$Y = 4\left(7n + 1 + 3(C-1) - 3\cdot\frac{C-1}{4}\right)$$

in which C = the century. The decimals must be neglected and n so chosen that the year falls within the century.

The years in which February has five of any other day of the week are given by the formula

$$Y = 4\left(7n + 3(W-1) + 1 + 3(C-1) - 3\cdot\frac{C-1}{4}\right)$$

in which W = the given day of the week.

The years next after leap years, or the inauguration years, in which March 4 comes on Sunday are given by the formula

$$Y = 4\left(7n + 1 + 3(C-1) - 3\cdot\frac{C-1}{4}\right) - 3.$$

Demonstration of two Diophantine Theorems.

By Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Pa.

1. The sum of the squares of three consecutive whole numbers can not be a square number.

Any three consecutive whole numbers may be represented by $x-1$, x and $x+1$; the sum of their squares is $3x^2+2$, which is of the form $3m+2$.

We must now show that no square number is of the form $3m+2$. All whole numbers are included in the forms $3n$, $3n+1$ and $3n+2$; and all square numbers are included in the squares of these forms. $(3n)^2 = 9n^2 = 3(3n^2)$; $(3n+1)^2 = 9n^2+6n+1 = 3(3n^2+2n)+1$, and $(3n+2)^2 = 9n^2+12n+4 = 3(3n^2+4n+1)+1$. All square numbers are therefore included in the forms $3m$, $3m+1$, and $3m+2$ can not be a square number.

Cor.—In a similar manner it may be shown that the sum of the squares of four consecutive numbers can not be a square.

2. The sum of five consecutive integral square numbers can not be a square number.

Let $x-2$, $x-1$, x, $x+1$, $x+2$ be the roots of the squares; then the sum of the squares is

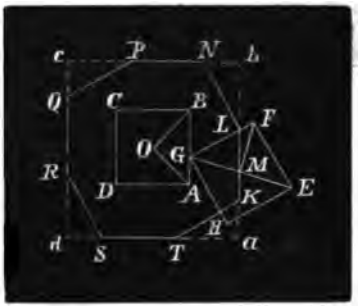
$$5x^2+10 = 5(x^2+2).$$

If this is a square the square must be divisible by 5. Put, therefore, $5(x^2+2) = 25y^2$; then $x^2+2 = 5y^2$, and x^2+2 must also be divisible by 5. All numbers divisible by 5 end with 0 or 5. All square numbers end with 0, 1, 4, 5, 6 or 9. Hence x^2+2 ends with 2, 3, 6, 7, 8 or 1 and therefore is never divisible by 5.

Solution of the "Three-Dice" Problem.

By E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Adair County, Missouri.

If three dice be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?



Let the squares ABCD and EFGH (Fig. 1) be the projections of the first and second dice on the horizontal plane, and O and M the centers of the squares. If the square EFGH be moved parallel to itself about the fixed square ABCD so as to be exterior to, but in contact with it, M will describe the octagon KLNPQRST whose area represents the number of positions the second die can have.

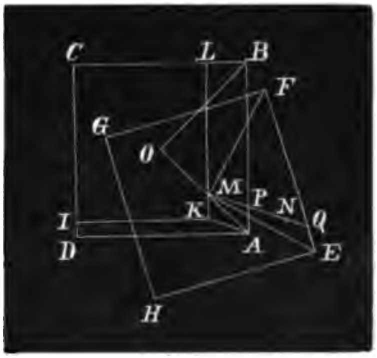
Let $AB = 1$, and let θ represent the angle AGH which is equal to the angle formed by AO and EG produced; then area KLNPQRST $=$ area $abcd - 4$ area $KaT = (1+\sin\theta+\cos\theta)^2 - 2\sin\theta\cos\theta$
$$= 2(1+\sin\theta+\cos\theta).$$

If ψ be the angle which a side of the third die makes with that of the second, the number of positions of the third die for each position of the second will be represented by $2(1+\sin\psi+\cos\psi)$.

Let N (Fig. 2) be the projection of the center of the third die.

The pile will not fall down, if M and the middle point of MN are in the square ABCD, and N is in the square EFGH.

If EFGH be moved parallel to itself so that MN will be constantly bisected by a side of ABCD, M will move over the line IKL, and the area of the rectangle IKLC will represent the number of favorable positions.



Let $MN = x$, $MQ = x'$, $\angle AME = \theta$, $\angle MQF = \varphi$, and area $CIKL = u$. Then we have $x' = \frac{1}{2}\operatorname{cosec}\varphi$; if φ is less than $\frac{1}{2}\pi - \theta$, the number of favorable positions is
$$u = [1 - \tfrac{1}{2}x\sin(\theta+\varphi)][1 - \tfrac{1}{2}x\cos(\theta+\varphi)],$$
and if φ is greater than $\frac{1}{2}\pi - \theta$, the number is
$$u_1 = [1 - \tfrac{1}{2}x\sin(\theta+\varphi)][1 + \tfrac{1}{2}x\cos(\theta+\varphi)].$$

The limits of θ are 0 and $\frac{1}{2}\pi$; those of φ, $\frac{1}{4}\pi$ and $\frac{1}{2}\pi - \theta$, $\frac{1}{2}\pi - \theta$ and $\frac{3}{4}\pi$; of x, 0 and x'; and of ψ, 0 and $\frac{1}{2}\pi$.

The results of the integrations with respect to φ must be multiplied by 4 to allow for the cases in which N falls on the other parts of EFGH.

Hence the required probability is

$$p = \frac{4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left\{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi-\theta}\int_0^{x'} u\,d\varphi x dx + \int_{\frac{1}{2}\pi-\theta}^{\frac{3}{4}\pi}\int_0^{x'} u_1\,d\varphi x dx\right\} d\theta d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} 4(1+\sin\theta+\cos\theta)(1+\sin\psi+\cos\psi)d\theta d\psi}$$

$$= \frac{\pi\int_0^{\frac{1}{2}\pi}\left\{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi-\theta}\int_0^{x'} u\,d\varphi x dx + \int_{\frac{1}{2}\pi-\theta}^{\frac{3}{4}\pi}\int_0^{x'} u_1\,d\varphi x dx\right\} d\theta}{\int_0^{\frac{1}{2}\pi}(\pi+4)(1+\sin\theta+\cos\theta)d\theta}$$

$$= \frac{4\pi}{(\pi+4)^2}\int_0^{\frac{1}{2}\pi}\left\{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi-\theta}\int_0^{x'} u\,d\varphi x dx + \int_{\frac{1}{2}\pi-\theta}^{\frac{3}{4}\pi}\int_0^{x'} u_1\,d\varphi x dx\right\} d\theta$$

$$= \frac{\pi}{192(\pi+4)^2}\int_0^{\frac{1}{2}\pi}\left\{\int_{\frac{1}{4}\pi}^{\frac{1}{2}\pi-\theta}(96 + 16\sin\theta - 16\cos\theta - 3\sin\theta\cos\theta - 16\sin\theta\cot\varphi - 16\cos\theta\cot\varphi\right.$$
$$+ 3\cos 2\theta\cot\varphi + 3\sin\theta\cos\theta\cot^2\varphi)\operatorname{cosec}^2\varphi\,d\varphi + \int_{\frac{1}{2}\pi-\theta}^{\frac{3}{4}\pi}(96 - 16\sin\theta - 16\cos\theta + 3\sin\theta\cos\theta - 16\sin\theta\cot\varphi$$
$$\left. + 16\cos\theta\cot\varphi - 3\cos 2\theta\cot\varphi - 3\sin\theta\cos\theta\cot^2\varphi)\operatorname{cosec}^2\varphi\,d\varphi\right\} d\theta$$

$$= \frac{\pi}{192(\pi+4)^2}\int_0^{\frac{1}{2}\pi}(192 - 32\cos\theta - 16\sec\theta + 2\cos 2\theta + \sec^2\theta)d\theta$$

$$= \frac{\pi}{(\pi+4)^2}\left[\frac{\pi}{4} + \frac{1}{96} - \frac{\sqrt{2}}{12} + \frac{1}{12}\log(\sqrt{2}-1)\right].$$

PROBLEMS.

231.—Proposed by P. F. Manoe, Alamos, Sonora, Mexico.

If from any point in a circular arc perpendiculars are drawn to the bounding radii, the distance of their feet is invariable.

232.—Proposed by Hon. Josiah H. Drummond, LL. D., Portland, Maine.

From the expressions $x^2+y^2-1=\Box$ and $x^2-y^2-1=\Box$ find a series of values for x and y in positive whole numbers, each pair of which shall be the initial terms of an infinite series of values. Also, the law of the latter series.

233.—Proposed by Dr. A. J. Purdie, Otselic, Chenango County, New York.

A cone is 30 inches in circumference at tne base and 120 feet in hight. What length of inch ribbon will wind around the cone, from bottom to top, leaving a space of 5 inches between?

234.—Proposed by Prof. W. P. Casey, C. E., San Francisco, California.

Let a, b, c be the three sides of a triangle, r, r', r'', r''' the radii of the four circles which touch them; prove by a geometrical demonstration that

$$ab+ac+bc=rr'+rr''+rr'''+r'r''+r'r'''+r''r'''.$$

235.—Proposed by De Volson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, New Jersey.

A surveyor at latitude 40° north sights due east along a true level at a vertical staff 3 miles distant; how far south of the parallel of latitude passing through the position of the surveyor will the foot of the staff be?

236.—Proposed by Wooster W. Beman, Assistant Professor of Mathematics, University of Michigan, Ann Arbor, Mich.

From the equations

$$\frac{x^2}{\mu^2}+\frac{y^2}{\mu^2-b^2}+\frac{z^2}{\mu^2-c^2}=1,\quad \frac{x^2}{\nu^2}+\frac{y^2}{\nu^2-b^2}+\frac{z^2}{\nu^2-c^2}=1,\quad \frac{x^2}{\rho^2}+\frac{y^2}{\rho^2-b^2}+\frac{z^2}{\rho^2-c^2}=1,$$

find the values of x, y and z by determinants.

237.—Proposed by D. J. McAdam, M. A., Professor of Mathematics, Washington and Jefferson College, Washington, Pa.

A street bears N. 12° W. At right angles to the street is a wall 22 feet high and 61 feet long. How long will the sun shine upon a shelf $4\frac{1}{2}$ feet high, at a point 27 feet from the street and 36 feet from the wall, on the 22d of December?

238.—Proposed by F. P. Matz, M. A., Professor of Pure and Applied Mathematics, Bowdon (State) College, Bowdon, Carroll County, Georgia.

At what angle with its axis must a vertical cylindric corn-stalk be struck with a sharp, wedge-shaped blade so as to sever the stalk with a blow of minimum force?

239.—Proposed by Dr. S. H. Wright, M. A., Ph. D., Mathematical Editor of the *Yates County Chronicle*, Penn Yan. Yates County, New York.

In what latitude will two stars cross the prime vertical at the same time, their declinations being δ and δ', and the difference of right ascension P?

240.—Proposed by W. W. Johnson, Member of the London Mathematical Society, Professor of Mathematics, St. John's College, Annapolis, Maryland.

The center of a circle which passes through a fixed point moves on a given curve; prove that the envelope is similar to the pedal of the given curve with reference to the fixed point.

241.—Proposed by Miss Christine Ladd, B. A., Baltimore, Maryland.

Given three circles; four pairs of circles may be drawn tangent to them, and four pairs of circles may be drawn through their six points of intersection. Show that the product of the lengths of tangents from the radical center of the given circles on any pair of tangent circles is equal to the product of the lengths of tangents from the same point on any pair of circles through the six (real or imaginary) points of intersection.

242.—Proposed by De Volson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, New Jersey.

A homogeneous sphere, mass m, radius r, having an angular velocity ω, is dropped; if the volume increases at a uniform rate, how far will it have descended when the angular velocity becomes ω'?

243.—Proposed by Reuben Davis, Bradford, Stark County, Illinois.

It is required to find three positive integral numbers, such that the sum of their cubes is a cube, and also the sum of the cubes of every two of them a cube.

244.—Proposed by C. A. O. Rosell, A. B., Teacher of German and Mathematics in the High School, Reading, Pa.

A cup has the volume v. What are its dimensions when its exterior surface s is a minimum?

245.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Two rods, lengths m and n, rest with their ends against each other in a fixed hemispheroidal bowl, radius of top a and depth b. Find the position of equilibrium.

246.—Proposed by CHARLES H. KUMMELL, Assistant Engineer, U. S. Lake Survey, Detroit, Michigan.

Find the equation and length of the shortest curve connecting two points on the helicoid, or winding-stair-case surface.

247.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A sportsman shot a duck in the middle of a river $2a$ yards wide, and his dog started after it when it was directly opposite him, swimming continually towards the duck. The velocity of the river is v miles an hour, and the dog can swim m miles an hour in still water. Required the equation to the *curve* the dog describes in space, the distance he swims to get the duck and the time occupied.

248.—Proposed by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics North Missouri State Normal School, Kirksville, Missouri.

Find the average area of the triangle formed by joining three points taken at random in the surface of a parabola whose base is b and altitude h.

249.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A point is taken at random in the surface of a given circle and two random chords drawn through it. Find the average area of the quadrilateral formed by joining the extremities of the chords.

250.—Proposed by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics in the North Missouri State Normal School, Kirksville, Missouri.

A plane triangle is formed by joining three points taken at random on the surface of a sphere; find (1) the average area of the triangle, and (2) the average area of its inscribed circle.

251.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Two coins, radii a and b, are lying *on* the bottom of a circular box, radius r, $>(a+b)$; find the chance that both are on the same diameter of the box.

252.—Proposed by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics North Missouri State Normal School, Kirksville, Adair County, Missouri.

A triangle is formed by joining three random points within a circle; find the chance that its circumscribing circle is less than the given circle.

253.—PRIZE PROBLEM. Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Pa.

A sportsman saw a duck in a river a yards wide which dodged behind a rock directly opposite him, b yards from the shore; he ran down stream, along the edge of the water, at a speed of m miles an hour, but the duck kept all the time "behind the rock" and swam for the "other shore". The velocity of the river is v miles an hour, and the duck can swim n miles an hour in still water. Required the equation to the *curve* the duck describes in space.

Solutions of these problems should be received by March 1, 1881.

EDITORIAL NOTES.

Professors W. W. JOHNSON of Annapolis, Md., and E. B. SEITZ of Kirksville, Mo., have been elected Members of the London Mathematical Society.

The Honorary Degree of Master of Arts has been conferred upon Professor F. P. MATZ by Ursinus College.

This No. of the VISITOR has been delayed some weeks by the ill health of the Editor, who has done all the type-setting himself; the press-work was done at the office of the Erie Gazette.

We hope some of our contributors will take more pains in the future in writing out their solutions; many of the errors which disfigure the pages of the present No. are *not* typographical. Read over your solutions several times after copying, and be *sure* they are "O. K." before sending them for publication.

Solutions of the problems proposed in No. 4 will be given in No. 6, and solutions of those proposed in this No. will be given in No. 7.

Contributors will oblige us by sending in their solutions of problems in No. 4 as soon as practicable.

Send subscriptions soon to ARTEMAS MARTIN, *Erie, Pa.*

NOTICES OF BOOKS AND PERIODICALS.

An Elementary Treatise on Analytic Geometry, Embracing Plane Geometry and an Introduction to Geometry of Three Dimensions. By Edward A. Bowser, Professor of Mathematics and Engineering in Rutgers College. 12mo, pp. 287. New York: D. Van Nostrand.

An excellent elementary work, containing a large number of well-selected examples, and very neatly printed.

Mathematical Questions, with their Solutions, from the "Educational Times", with many Papers and Solutions not published in the "Educational Times". Edited by W. J. C. Miller, B. A., Registrar of the General Medical Council. Vol. XXXII. From July to December, 1879. 8vo, boards, pp. 112. London: C. F. Hodgson & Son.

This interesting volume contains 4 papers and solutions of 111 problems. Several of the most difficult Probability solutions are by our valued contributor, Professor Seitz.

The "Educational Times" is published monthly; the *Reprint*, half-yearly. Vol. XXXIII is announced as nearly ready.

The Editor of the VISITOR can furnish the "Times" at $2.00 a year, and the *Reprint* at $1.75 per volume.

How to Secure and Retain Attention. By James L. Hughes, Inspector of Public Schools, Toronto, Canada. 18mo, pp. 86. Toronto: W. J. Gage & Co.

An admirable little book that should be read by every teacher.

American Journal of Mathematics Pure and Applied. Editor in Chief: J. J. Sylvester. Associate Editor in Charge: William E. Story. With the Co-operation of Simon Newcomb, H. A. Newton, and H. A. Rowland. Published under the Auspices of the Johns Hopkins University. Baltimore: Printed for the Editors by John Murphy & Co. Quarterly; 4to, pp. 100. $5.00 a year; single numbers, $1.50. Vol. II.

No. I, Vol. II, contains a very interesting and valuable paper on the Pascal Hexagram by Miss Christine Ladd. This volume also contains many other articles of special interest, but we have not room to mention even their titles.

The world-wide reputation of the distinguished Editors is a sufficient guaranty that the pages of the *Journal* will always be filled with matter of the highest order.

The Analyst: A Journal of Pure and Applied Mathematics. Edited and Published by J. E. Hendricks, M. A., Des Moines, Iowa. Bi-monthly; 8vo, pp. 32. $2.00 a year. Vol. VII.

Devoted to valuable mathematical papers and solutions of interesting problems, is ably edited and should be liberally supported.

The School Visitor, Devoted to the Study of Mathematics and English Grammar. Edited by J. S. Royer and Thomas Ewbank. Monthly; pp. 12. $0.60 a year; single numbers, 10 cents. Ansonia, O: Published by John S. Royer.

This handsome youngster increases in interest with its age. The July No. contains 7 pages of excellent solutions in elementary mathematics, illustrated with beautiful diagrams, and a list of problems for solution in future numbers, besides much other matter of general interest to teachers. We wish it abundant success.

The Wittenberger. 10 numbers in a year. $1.00 a year. Springfield, O: Wittenberg College.

The Mathematical Department is efficiently conducted by William Hoover, Superintendent of Schools, Wapakoneta, O.

Barnes' Educational Monthly. New York: A. S. Barnes & Co. $1.50 a year,

The Mathematical Department is ably edited by Prof. F. P. Matz, M. A., of Bowdon College, Ga.

Educational Notes & Queries. Edited and Published by Prof. W. D. Henkle, Salem, O. $1.00 a year.

Ten numbers in a year. Contains, besides other matter of general interest, mathematical notes and problems and solutions.

The Yates County Chronicle. Weekly. Penn Yan, N. Y.: Chronicle Publishing Co. $2.00 a year.

The Mathematical Department is ably edited by Dr. S. H. Wright, M. A., Ph. D.

The Canada School Journal. Monthly. Toronto, Canada: W. J. Gage & Co. $1.00 a year in advance.

Has an excellent Mathematical Department edited by Alfred Baker, M. A.

New-England Journal of Education. Thomas W. Bicknell, Editor. Boston: 16 Hawley street. $3.00 a year; $2.50 in advance.

The Pennsylvania School Journal. Monthly. J. P. Wickersham, Editor. Lancaster: J. P. Wickersham & Co. $1.60 a year.

American Journal of Education. Monthly. Saint Louis: J. B. Merwin. $1.60 a year.

CORRIGENDA.

No. 2.

Page 28, line 27, *for* "4mna" *read* $4m^2a$.

No. 3.

Page 63, solution of Problem 59, line 5, *for* "$\lambda = \sin^{-1}(\sin A \sin\delta)$" *read* $\lambda = \sin^{-1}(\sin A \cos\delta)$.

No. 4.

Page 92, solution of Problem 101, the letter S is omitted in the diagram.
Page 108, solution of Problem 127, next to the last line, *for* "$\sqrt{[x^2+y^2-2xy\cos\omega]}$" *read* $\sqrt{[x^2+y^2+2xy\cos\theta]}$.
Page 110, solution of Problem 132, the letter P is omitted in the diagram.
Page 111, solution of Problem 133, in the denominator of the value of t, at the end of the last line *for* "v" *read* v^2.
Page 116, Problem 199, Prof. Barbour's address should be Richmond, Madison Co., Ky.

No. 5.

Page 126, second solution of Problem 114, line 3, *for* "(3)" *read* (4), and *for* "$= \square$" *read* $= m^2$; line 5, numerator of the value of x, *for* "$(m-1)$" *read* $(p-1)$.

Page 127, line 19, the subscript 8's should be 9's; in the values of the edges of the 301st solid insert a "4" between the 23d and 24th figures.

Page 128, in the value of the 301st diagonal insert an "8" between the 33d and 32d figures from the end of the line.

Page 129, solution of Problem 116, line 9, *for* "$6\phi^2$" *read* $b\phi^2$; and *for* "$30a^2c^2$" *read* $3a^2c^2$.

Page 133, second solution of Problem 129, line 2, *for* "$x^3+y^3+z^3, -$" *read* $x^3+y^3+z^3+3, =$.

Page 137, line 5, after "$d\theta$" insert "dx".

Page 138, line 7, denominator of last integral, *for* "dx" *read* $\wedge$; last line, *for* "Δe" *read* Δa.

Page 142, line 4, *for* "CN′" *read* CN. The letters D and S are omitted in the diagram, and "G" is put for Q.

Page 143, "E" is omitted in the diagram.

Page 144, 6th integration, in the denominator of the first integral, *for* "π^4" *read* π^3, and in the denominator of the second integral *for* "r^3" *read* π^3; in the last integration, first integral, the factor $\sin^4\theta$ outside the parenthesis should be $\sin^6\theta$. For the second integral substitute the following:

$$\frac{8r^3}{81\pi^3}\int_0^{\pi}(33\theta - 54\theta^2\sin\theta\cos\theta + 72\theta\cos^2\theta - 48\sin^7\theta\cos\theta - 2\sin^5\theta\cos\theta + 26\sin^3\theta\cos\theta$$
$$- 15\sin\theta\cos\theta - 48\sin^5\theta\cos^3\theta - 60\sin^3\theta\cos^3\theta - 90\sin\theta\cos^3\theta)\sin^4\theta\, d\theta.$$

Page 145, at the end of line 2 of the solution insert "inscribed circle of the polygon, and M the center of the".

Page 147, line 7, *for* "$\frac{1}{16}p[\frac{1}{4}p(1-e^2)-\frac{1}{4}x]$" *read* $\frac{1}{16}p[\frac{1}{4}p(1-e^2)-\frac{1}{4}x]\sqrt{(p+x)}\sqrt{(e^2p-x)}$.

Page 152, first line below the second brace, insert "+" before the last term.
Page 154, line 8 from the bottom, after e insert "$\pm f$".
Page 155, line 25, *for* "-3380" *read* -3360.

Vol. I. JANUARY, 1881. No. 6.

THE MATHEMATICAL VISITOR.

EDITED AND PUBLISHED BY

ARTEMAS MARTIN, M. A.,

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1881.

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Vol. I. *JANUARY, 1881.* *No. 6.*

JUNIOR DEPARTMENT.

Solutions of the Problems Proposed in No. 4.

148.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A man bought a horse for $100, and sold him for $60, and then bought him back for $75. How much did he lose by the transaction?

I.—Solution by J. M. QUIROZ, Alamos, Sonora, Mexico; JOSEPH TURNBULL, East Liverpool, Ohio; EDWIN PLACE, Cincinnatus, New York; and J. W. DONOVAN, Ansonia, Ohio.

If he sold a horse which cost him $100 for $60, the man lost $40. By buying the same horse again for $75 he gained $25. Therefore the difference between $40 and $25, or $15, is what he lost.

II.—Solution by Dr. DAVID S. HART, M. A., Stonington, Connecticut; and L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.

If the man had repurchased the horse at the same price for which he sold it, he would have lost nothing. Therefore he lost the difference between the two prices; that is, 75 — 60 = 15 dollars, his loss.

III.—Solution by Professor W. P. CASEY, C. E., San Francisco, California.

As the man had to add only $15 to the $60 to make the second purchase, he therfore lost only $15.

IV.—Solution by ORLANDO D. OATHOUT, Luana, Iowa; and Hon. J. H. DRUMMOND, LL. D., Portland, Maine.

Having the horse, no matter what it cost, the man sold it for $60, and bought it back for $75; therefore he lost $75 — $60 = $15.

V.—Solution by H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, North Carolina.

By selling and then buying back again the horse stands him 100 — 60 + 75 = 115 dollars; 115 — 100 = 15 dollars, his loss.

Good solutions from Messrs. *Brown*, *Clark*, *Hoover*, *Nichols*, *Pollard*, *Sylvester Robins*, *C. C. Robins*, *Rosell* and *Siverly*.

149.—Proposed by JOHN I. CLARK, Moran, Clinton County, Indiana.

My grocer sold me a cheese, which he said weighed 32 pounds; but when it was placed on the other side of the scales, it only weighed 18 pounds. He then proposed that I should buy another of the same size, and weigh it on the opposite side from the first, to which I consented. Did I gain or lose by the transaction? And what was the true weight of the cheese?

Solution by A. E. HAYNES, Professor of Mathematics, Hillsdale College, Hillsdale, Michigan; C. A. O. ROSELL, A. B., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; J. M. TAYLOR, Milton, Oregon; CHARLES GILPIN, Jr., Philadelphia, Pa.; and C. C. ROBINS, Princeton, New Jersey.

Let x = the true weight of the cheese, a and b the false weights, and c and e the arms of the scales. Then, from the principles of Mechanics, we have the equations: $cx = be$ ...... (1), $ex = ac$ ...... (2).

Multiplying (1) by (2), $cex^2 = abce$; $\therefore$ $x^2 = ab$, and $x = \sqrt{(ab)}$.

Therefore the true weight is a mean proportional between the false weights, and the weight of each cheese was 24 lbs.; and since the purchaser paid for $32+18=50$ lbs. and obtained only $24\times 2=48$ lbs., he lost the price of 2 lbs.

The problem was also solved by Messrs. *Bagot, Casey, Clark, Donovan, Hawley, Hoover, Ludwig, Matz, Nichols, Oathout, Pollard, Sylvester Robins, Seitz* and *Siverly*.

150.—Proposed by W. S. HINKLE, Leacock, Lancaster County, Pennsylvania.

How much more can a bank make in 528 days, with \$10000, by discounting notes on 45 days time, than by discounting them on 30 days, the rate of discount being 6 per cent., and the profits in both cases retained in bank till the expiration of the time?

Solution by JULIAN A. POLLARD, Goshen, N. Y.; C. A. O. ROSELL, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; SYLVESTER ROBINS, North Branch Depot, N. J.; and C. C. ROBINS, Princeton, N. J.

By discounting notes on 45 days time, the bank could make $528\div(45+3)=11$ transactions; and by discounting them on 30 days time, $528\div(30+3)=16$ transactions.

In the first case the discount, at 6 per cent. per annum, is $\$0.06\times\frac{48}{360}=\$0.008$ on the dollar. The profit of each transaction would be $\frac{\$10000}{1-0.008}-\$10000=\$80.645161294+$, and of the 11, 11 times as much, or \$887.096774.

In the second case, the discount is $\$0.06\times\frac{33}{360}=\$0.0055$ on the dollar. The profit of each transaction would be $\frac{\$10000}{1-0.0055}-\$10000=\$55.3041729+$, and on the 16, $\$55.3041729\times 16=\$884.866766$.

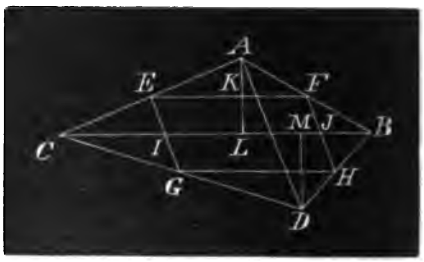
$\$887.096774-\$884.866766=\$2.23$, what the bank would gain by discounting on 45 days time.

Solved in a similar manner by Messrs. *Clark, Donovan, Drummond, Matz, Seitz* and *Siverly*.

151.—Proposed by G. F. MEAD, Uniontown, Fayette County, Pennsylvania.

If the sides of *any* quadrilateral be bisected, and the points of bisection joined, the resulting figure will be a parallelogram and equal in area to half the quadrilateral.

Solution by J. M. QUIROZ, Alamos, Sonora, Mexico; WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Ohio; C. A. O. ROSELL, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; Prof. W. P. CASEY, C. E., San Francisco, California; P. F. MANGE, Alamos, Sonora, Mexico; SYLVESTER ROBINS; JULIAN A. POLLARD; EDWIN PLACE; and ORLANDO D. OATHOUT.



Let ABCD be any quadrilateral, E, F, G, H the middle points of the sides. Join E and F, E and G, F and H, and G and H. Draw the diagonals AD, BC.

In the triangle ABD the sides AB, BD are bisected in F and H, therefore FH is parallel to AD; in the triangle ACD the sides AC, CD are bisected in E and G, therefore EG is parallel to AD, and also parallel to FH.

In like manner it is shown that EF is parallel to GH, and therefore EFGH is a parallelogram.

Draw AL and DM perpendicular to CB. Area of triangle $ACB=AL\times\frac{1}{2}BC$, area of parallelogram $EFIJ=KL\times EF=\frac{1}{2}AL\times\frac{1}{2}BC$; therefore parallelogram $EFIJ=\frac{1}{4}$ of triangle ACB.

Area of triangle $DCB=DM\times\frac{1}{2}BC$, area of parallelogram $GHIJ=MN\times GH=\frac{1}{2}DM\times\frac{1}{2}BC$; therefore parallelogram $GHIJ=\frac{1}{4}$ of triangle DCB.

Adding, we get, parallelogram $EFGH=\frac{1}{2}$ of quadrilateral ABCD.

Solved also by Messrs. *Clark, Donovan, Drummond, Hart, Hawley, Ludwig, Matz, Nichols, Putnam, C. C. Robins, Schaffer, Seitz, Siverly* and *Turnbull*.

152.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A company of n men were counting their money. The first said to the second, "Give me your money and I will have \$$a$"; the second said to the third, "Give me $\frac{1}{2}$ of yours and I will have \$$a$; the third said to the fourth, "Give me $\frac{1}{3}$ of yours and I will have \$$a$"; ...... the nth said to the first, "Give me one-nth of yours and I will have \$$a$". What sum had each?

I.—Solution by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; and SYLVESTER ROBINS, North Branch Depot, N. J.

Let $x=$ amount the first man had; then $a-x=$ amount second had, $2[a-(a-x)]=2x=$ amount third had, $3(a-2x)=3a-2.3x=$ amount fourth had, $4[a-3(a-2x)]=4a-3.4a+2.3.4x=$ what fifth had, $5[a-(4a-3.4a+2.3.4x)] = 5a-4.5a+3.4.5a-2.3.4.5x$ = amount sixth had; from which it is evident that the nth man had

$$(n-1)a-(n-1)(n-2)a+(n-1)(n-2)(n-3)a- \ldots\ldots \pm 1.2.3.4\ldots(n-1)x,$$

which by the terms of the question $= a - \frac{x}{n}$. Reducing this equation we readily find

$$x = \frac{a[n-n(n-1)+n(n-1)(n-2)-n(n-1)(n-2)(n-3)+\ldots\ldots \pm 3.4.5\ldots n]}{1 \pm 1.2.3.4\ldots n}.$$

The upper signs obtain when n is odd and the lower when n is even.

II.—Solution by Lucius Brown, Hudson, Massachusetts; J. F. W. Scheffer, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; C. A. O. Roeell, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; and Walter S. Nichols, Editor *Insurance Monitor*, New York, N. Y.

Let $x_1, x_2, x_3, \ldots\ldots x_n$ denote the several sums. Then by the given conditions we have

$$x_1+x_2 = a\ldots\ldots(1),\quad x_2+\tfrac{1}{2}x_3 = a\ldots\ldots(2),\quad x_3+\tfrac{1}{3}x_4 = a\ldots\ldots(3),\quad x_4+\tfrac{1}{4}x_5 = a\ldots\ldots(4),$$

$$\ldots\ldots\ldots\ldots x_{n-1}+\frac{1}{n-1}x_n = a\ldots\ldots(n-1),\quad x_n+\frac{1}{n}x_1 = a\ldots\ldots(n).$$

Multiply the first, second, third, fourth, ... $(n-1)$th equations by $1, -1, +\frac{1}{1.2}, -\frac{1}{1.2.3}, \ldots\ldots \pm\frac{1}{1.2.3\ldots(n-2)}$, and add the results, and we get

$$x_1+x_2-x_2-\tfrac{1}{2}x_3+\tfrac{1}{2}x_3+\tfrac{1}{6}x_4-\tfrac{1}{6}x_4-\tfrac{1}{24}x_5+\ldots\ldots \pm\frac{1}{1.2.3\ldots(n-2)}\left(x_{n-1}+\frac{1}{n-1}x_n\right) =$$

$$a\left(1-1+\frac{1}{1.2}-\frac{1}{1.2.3}+\frac{1}{1.2.3.4}-\frac{1}{1.2.3.4.5}+\ldots\ldots\pm\frac{1}{1.2.3\ldots(n-2)}\right);$$

or

$$x_1 \pm \frac{x_n}{1.2.3\ldots(n-1)} = aS_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(n+1),$$

where S_{n-1} is put for the series on the right, which is the reciprocal of the base of the Naperian system of logarithms to $n-1$ terms.

From (n) and $(n+1)$ we readily find $x_1 = \frac{a[(1.2.3\ldots n)S_{n-1} \pm n]}{1.2.3\ldots n \pm 1}$, $x_n = \frac{a(1.2.3\ldots n)(n-S_{n-1})}{n(1.2.3\ldots n\pm 1)}$.

III.—Solution by Walter Siverly, Oil City, Venango County, Pennsylvania; and William Hoover, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.

Let $u_1, u_2, u_3, \ldots\ldots u_x, \ldots\ldots u_n$ be the sums the first, second, third, ... xth, ... and nth persons have respectively.

Then $u_x + \frac{u_{x+1}}{x} = a$, which gives the difference equation

$$u_{x+1}+xu_x = ax\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Integrating,

$$u_x = (-1)^{x-1}\left[(1.2.3\ldots x-1)\left(C-\frac{1}{1}a+\frac{2}{1.2}a-\frac{3}{1.2.3}a+\ldots\ldots+(-1)^{x-1}\cdot\frac{x-1}{1.2.3\ldots x-1}a\right)\right]$$

$$= (-1)^{x-1}\left[(1.2.3\ldots x-1)\left(C-\frac{1}{1}a+\frac{1}{1}a-\frac{1}{1.2}a+\frac{1}{1.2.3}a-\ldots\ldots\pm\frac{1}{1.2.3\ldots x-2}a\right)\right]$$

$$= (-1)^{x-1}(1.2.3\ldots x-1)(C-aS_{x-1}).$$

When $x=1$, $C=u_1$;

$$\therefore u_x = (-1)^{x-1}(1.2.3\ldots x-1)(u_1-aS_{x-1})\ldots\ldots(2),\quad u_n = (-1)^{n-1}(1.2.3\ldots n-1)(u_1-aS_{n-1})\ldots\ldots(3).$$

By the problem, $u_n+\frac{u_1}{n} = a\ldots\ldots(4)$. Eliminating u_n from (3), (4), $u_1 = \frac{a[(1.2.3\ldots n)S_{n-1}\pm n]}{1.2.3\ldots n\pm 1}$.

Substituting this value of u_1 in (2) and (3) we get

$$u_x = a(1.2.3\ldots x-1)\left(\frac{(1.2.3\ldots n)S_{n-1}\pm n}{1.2.3\ldots n\pm 1}-S_{x-1}\right),\qquad u_n = \frac{a(1.2.3\ldots n)(n-S_{n-1})}{n(1.2.3\ldots n\pm 1)}.$$

Solved also in an elegant manner by Messrs. *Casey, Donovan, Matz, Pollard* and *Putnam*.

153.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

What are the first ten numbers which, when multiplied by 1, 2, 3, 4, etc., will give products containing the same figures, and in the same order though beginning at a different digit, but when multiplied by 7, 17, —, —, etc., respectively, will give products containing all nines?

Solution by the PROPOSER.

Dividing a number consisting of 9's by 7 we find the quotient terminates with six places, and is 142857. This is the first number, and when multiplied by 1, 2, 3, 4, 5 and 6, respectively, gives products containing the same figures, and in the same order but beginning at different digits. The second number is obtained by dividing a number composed of 9's by 17. This time the division terminates with 16th 9 used, giving as a quotient 5882352941176470. Multiplied by the numbers 1 to 16 inclusive, the several products are seen to observe the same law as before.

"*Every prime number except* 2 *and* 5 *will exactly divide the number expressed by as many 9's as there are units, less one, in the divisor.*"—LEGENDRE.

When the division does not terminate before reaching the last of the dividend there will stand in the middle place or places of the quotient as many 9's as there are figures in the divisor, less 1; and all the figures in the latter part of the quotient may be found by subtracting those in the former half thereof successively from 9's; and the entire quotient will possess the remarkable property of producing other numbers containing the same figures as itself, in exactly the same order although beginning at a different digit, when multiplied by any number less than said prime divisor.

The other numbers result from the division of numbers composed of 9's by 19, 23, 29, 47, 59, 61, 97, 109, 113, 131, 149, &c., and are:

3. 52631578947368421; **4.** 4347826086956521739 13; **5.** 3448275862068965517241379 31;

6. 212765957446808510638297872340425531914893617;

7. 16949152542372881355932203389830508474576271186440677 9661;

8. 16393442622950819672131147540983606557377049180327868852459;

9. 1030927835051546391752577319587628865979381443298969072164948453608247422680412371134020618556 7;

10. 917431192660550458715596330275229357798165137614678899082568807339449541284403669724770642201834 8623853211;

11. 884955752212389380530973451327433628318584070796460176991150442477876106194690265486725663716814 15929203539823;

12. 763358778625954198473282442748091603053435114503816793893129770992366412213740458015267175572519 08396946564885496183206106870229;

13. 671140939597315436241610738255033557046979865771812080536912751677852348993288590604026845637583 89261744966442953020134228187919463087248322147651; etc.; etc.

Excellent solutions received from Professor *Scheffer*, Professor *Rosell* and *Walter Siverly*.

154.—Proposed by JOSEPH FICKLIN, M. A., Ph. D., Professor of Mathematics and Astronomy, University of the State of Missouri, Columbia, Boone County, Missouri.

The area of a triangle ABC is b; the side AB is a, and the angle opposite AB is β. Required the sides AC and BC.

Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin Co., Pa.; L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; and WALTER SIVERLY, Oil City, Venango Co., Pa.

Denoting the side AC by x and BC by y, we have

$$xy\sin\beta = 2b \;\ldots\ldots\ldots\ldots (1), \qquad x^2+y^2-2xy\cos\beta = a^2 \;\ldots\ldots\ldots\ldots (2).$$

Substituting $xy = 2b\operatorname{cosec}\beta$ in (2), we have

$$x^2+y^2 = a^2+4b\cot\beta.$$

Adding $2xy = 4b\operatorname{cosec}\beta$ to, and also subtracting it from, the last equation we obtain

$$x+y = \sqrt{(a^2+4b\cot\tfrac{1}{2}\beta)}, \qquad x-y = \sqrt{(a^2-4b\tan\tfrac{1}{2}\beta)};$$

$$\therefore\; x = \tfrac{1}{2}\sqrt{(a^2+4b\cot\tfrac{1}{2}\beta)}+\tfrac{1}{2}\sqrt{(a^2-4b\tan\tfrac{1}{2}\beta)}, \quad y = \tfrac{1}{2}\sqrt{(a^2+4b\cot\tfrac{1}{2}\beta)}-\tfrac{1}{2}\sqrt{(a^2-4b\tan\tfrac{1}{2}\beta)}.$$

Solved also by Messrs. *Banks*, *Casey*, *Drummond*, *Hart*, *Hoover*, *Ludwig*, *Matz*, *Oathout*, *Putnam*, *Rosell*, *C. C. Robins*, *Seitz* and *Turnbull*.

155.—Proposed by G. F. Mead, Uniontown, Fayette County, Pennsylvania.

If from any point in a diagonal of a parallelogram lines be drawn to the opposite angles the parallelogram will be divided into two pairs of equivalent triangles.

Solution by P. F. Mange, Alamos, Sonora, Mexico; E. J. Edmunds, B. S., Ancien Eleve de l'Ecole Polytechnique, Professeur de Français d'Anglais et de Mathematiques, Paris, France; H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C.; Prof. W. P. Casey, C. E., San Francisco, California; Edwin Place, Cincinnatus, N. Y.; George Hawley, San Francisco, California; and K. S. Putnam, Rome, N. Y.

Let ABCD be a parallelogram, AC a diagonal and P any point in it. Join PB and PD. Draw BH and DK perpendicular to AC. As the triangles ABC, ADC are halves of the same parallelogram, they are equal in all their parts; therefore BH = DK. The triangles APB and APD have the same base and equal altitudes, therefore they are equivalent. By the same method of reasoning the triangles CPB, CPD are shown to be equivalent.

Similar solutions given by Messrs. *Banks*, *Clark*, *Drummond*, *Hart*, *Haynes*, *Hoover*, *Matz*, *Oathout*, *Pollard*, *Quiroz*, *Sylvester Robins*, *C. C. Robins*, *Rosell*, *Scheffer*, *Seitz*, *Shidy* and *Siverly*.

156.—Proposed by Theodore L. DeLand, Office of the Secretary of the Treasury, Washington, D. C.

A owes B \$1750, payable as follows: 70 notes of \$25 each, the first payable in one month, the second in two, and so on to the last which is payable in seventy months; each bears simple interest at 10 per cent. per annum payable with the note. When will he neither gain nor lose by borrowing money at 8 per cent. per annum simple interest to pay all the unpaid notes and interest, and give new notes, interest due added, payable, the first one month from the date of the change, the second two months, and the last on the date when the original notes terminated?

I.—Solution by J. W. Donovan, Ansonia, Darke County, Ohio; and J. A. Pollard, Goshen, Orange County, N. Y.

The interest on \$25 for one year at 10 per cent. is \$2.50. The notes must run a sufficient length of time that the interest at 8 per cent. on the amount will be \$2.50. \$2.50 ÷ 0.08 = \$31.25, principal at 8 per cent. \$31.25 — \$25 = \$6.25, interest accrued at 10 per cent. If it requires one year to gain \$2.50 interest, to gain \$6.25 will require 2 years and 6 months, or 30 months, from date of first notes.

II.—Solution by John I. Clark, Moran, Clinton County, Indiana.

The last note, which will be due in 70 months, is sufficient for our purpose. If we find the time at which we can borrow money at 8 per cent. and pay the last note of \$25 with interest, and neither gain nor lose, it will be the time for all the other notes.

We first want to find the amount of money that will accumulate the same amount of interest at 8 per cent. that \$25 will at 10 per cent. in the same time, thus:

8 : 10 :: \$25 : \$31.25.

Now, if \$25 will amount to \$31.25 at 10 per cent. in a certain time, then that is the time we wish to find. \$25 will gain \$2.50 in 12 months at 10 per cent.; then

\$2.50 : \$6.25 :: 12 months : 30 months,

which is the time required.

Solved algebraically by the *Proposer*, and Messrs. *Casey*, *Putnam*, *Rosell*, *Seitz* and *Siverly*.

157.—Proposed by Sylvester Robins, North Branch Depot, Somerset County, New Jersey.

It is required to prove that a polynomial of n terms may be found which can be divided by a polynomial of n terms and give a quotient of n terms, the coefficient of every term in all of them being unity.

Solution by F. P. Matz, M. A., Professor of Mathematics, Military and Mathematical School, King's Mountain, N. C.

There may be two cases:—I, n may be odd; II, n may be even.

I. We readily perceive that

$$(x^2+x+1)(x^2-x+1) = x^4+x^2+1,$$

$$(x^4+x^3+x^2+x+1)(x^4-x^3+x^2-x+1) = x^8+x^6+x^4+x^2+1,$$

. . . .

$$(x^{n-1}+x^{n-2}+x^{n-3}+\dots+1)(x^{n-1}-x^{n-2}+x^{n-3}-\dots+1)$$
$$= x^{2(n-1)}+x^{2(n-2)}+x^{2(n-3)}+\dots+1 \quad \text{.........(A).}$$

Either factor of the first member of (A) will divide the second member in accordance with the requirements of the problem.

II. In this case,

$$(x+1)(x-1) = x^2-1, \qquad (x^3+x^2+x+1)(x^3-x^2+x-1) = x^6+x^4-x^2-1,$$
$$(x^5+x^4+x^3+x^2+x+1)(x^5-x^4+x^3-x^2+x-1) = x^{10}+x^8+x^6-x^4-x^2-1,$$

....

$$(x^{n-1}+x^{n-2}+x^{n-3}+\ldots\ldots+1)(x^{n-1}-x^{n-2}+x^{n-3}-\ldots\ldots-1)$$
$$= x^{2(n-1)}+x^{2(n-2)}+x^{2(n-3)}+\ldots\ldots+x^n-x^{n-2}-x^{n-4}-\ldots\ldots-1 \ldots\ldots\ldots (B).$$

Each factor of the first member of (B) will divide the second member.

Solved also by the *Proposer*, Messrs. *Drummond*, *Hart*, *Putnam*, *Rosell*, *Scheffer*, *Seitz* and *Siverly*.

158.—Proposed by G. F. MEAD, Uniontown, Fayette County, Pennsylvania.

If two circles intersect, the common chord produced will bisect the common tangent.

I.—Solution by EDWIN PLACE, Cincinnatus, Cortland County, N. Y.; J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; and WALTER SIVERLY, Oil City, Pa.

Let O and O′ be the centers of two intersecting circles, PP′ their common tangent, BS their common chord AS produced to meet it. From the point P′ draw P′T parallel to OO′ the line passing through the centers of the circles, and from P and P′ drop perpendiculars to the line through O and O′. Call $OP = R$, $OP' = r$ and $OO' = a$.

From the similar triangles P′TP, POE, P′O′F we have

$$\frac{PT}{P'T} = \frac{OE}{OP} = \frac{O'F}{O'P'}; \text{ or } \frac{R-r}{a} = \frac{OE}{R} = \frac{O'F}{r};$$

therefore $$OE = \frac{R(R-r)}{a} \text{ and } O'F = \frac{r(R-r)}{a}.$$

In the triangle AOO′ we have $(OA)^2-(OC)^2 = (AC)^2 = (AO')^2-(CO')^2$;

$$\therefore\ (OC)^2-(CO')^2 = (OA)^2-(AO')^2 = R^2-r^2 \ldots\ldots (1). \quad \text{Also, } OC+CO' = a \ldots\ldots (2).$$

Dividing (1) by (2), $OC-CO' = \dfrac{R^2-r^2}{a} \ldots\ldots (3)$. From (2) and (3) we easily find

$$OC = \tfrac{1}{2}a+\frac{R^2-r^2}{2a} = \frac{a^2+R^2-r^2}{2a}, \qquad CO' = \tfrac{1}{2}a-\frac{R^2-r^2}{2a} = \frac{a^2-R^2+r^2}{2a}.$$

$$EC = OC-OE = \frac{a^2+R^2-r^2}{2a}-\frac{R(R-r)}{a} = \frac{a^2-(R-r)^2}{2a},$$

$$CF = CO'+OF = \frac{a^2-R^2+r^2}{2a}+\frac{r(R-r)}{a} = \frac{a^2-(R-r)^2}{a}.$$

$\therefore$ EC = CF and consequently PB = BP′.

II.—Solution by C. C. ROBINS, Princeton, New Jersey; GEORGE HAWLEY, San Francisco, California; J. M. QUIROZ, Alamos, Sonora, Mexico; C. A. O. ROSELL, A. B., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; and WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Ohio.

$BA \times BS = (PB)^2$ and $BA \times BS = (BP')^2$, therefore $(PB)^2 = (BP')^2$, PB = BP′ and PP′ is bisected by AS produced.

Solved in a similar manner by Messrs. *Putnam*, *Sylvester Robins*, *Seitz*, *Shaw* and *Shidy;* and otherwise by Messrs. *Edmunds*, *Hart*, *Haynes*, *Mange*, *Matz*, *Siverly* and *Turnbull*.

159.—Proposed by WILLIAM HOOVER, Superintendent, of Schools, Wapakoneta, Auglaize County, Ohio.

August, 1879, had five Fridays, Saturdays and Sundays; when will the month of August have five of each of these days again?

I.—Solution by JOHN I. CLARK. Moran, Clinton County, Indiana; C. A. O. ROSELL, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa.; Dr. DAVID S. HART, M. A., Stonington, Conn.; and J. A. POLLARD, Goshen, N. Y.

In order that August can have five Fridays, Saturdays and Sundays, the first day must be Friday and the last Sunday. This would take place every seven years if there were no leap years, because there are 52 weeks and one day in a common year, consequently the first day of August will come one day later in the week each year; but it will come 2 days later each leap year, and as there will two leap years intervene between 1879 and 1886, said event will take place two years sooner, or in 1884.

Nearly thus were the solutions by *L. P. Shidy*, *C. C. Robins* and *Sylvester Robins*.

II.—Solution by the PROPOSER; and WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Any year in which the first of August falls on Friday will satisfy the problem.

The Dominical Letter for Friday, August 1, 1879, is E, and the remaining years of this century having E for their Dominical Letter are 1884 and 1890, the years required.

III.—Solution by K. S. PUTNAM, Rome, Oneida Co., N. Y.; and ELIJAH A. SQUIER, Le Grand, Marshall Co., Iowa.

The month of August will have five Fridays, Saturdays and Sundays in any year when August 1st comes on Friday.

To ascertain when the same day of any month in any given year will next come on the same day of the week, divide the last two figures by 4 and note the remainder. For January and February if the remainder is 3 it will occur 11 years afterwards; if the remainder is 1 or 2, 6 years; if remainder is 0, 5 years afterwards. For other months, if the remainder is 3 it will occur in 5 years; if 2, in 11 years; and if 1 or 0, in 6 years.

Should any centennial year not a leap year (as 1900) intervene, add 1 to the result obtained above.

In the problem given, 79 divided by 4 gives 3 as remainder, and the year required is 1884.

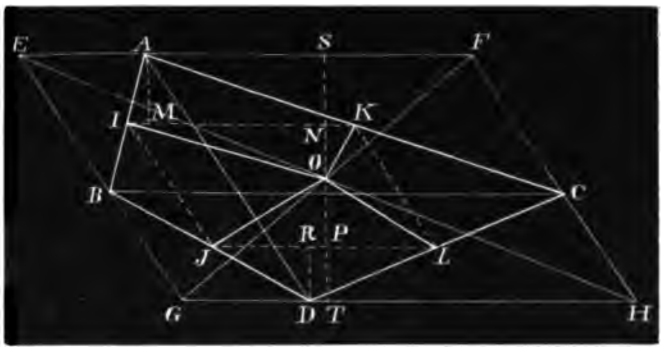
Following the rules the same will occur in 1890, 1902, 1913, 1919, 1924, 1930, 1941, 1947, 1952, 1958, 1969, 1975, 1980, &c.

An elaborate general solution was furnished by Prof. *Frank T. Freeland*, of the University of Pennsylvania, which we reluctantly omit for want of space.

160.—Proposed by L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.

Find a point within a given quadrilateral, such that when joined to the middle points of the sides the quadrilateral will be divided into four equivalent parts.

I.—Solution by K. S. PUTNAM, Rome, Oneida County, N. Y.; and JOHN I. CLARK, Moran, Clinton County, Indiana.



Let ABCD be any quadrilateral. Through A and D respectively draw EF and GH parallel to the diagonal BC; through B and C, EG and FH parallel to the diagonal AD. O, the intersection of the diagonals of the circumscribed parallelogram, will be the point required.

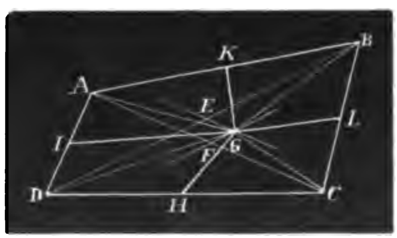
I, J, K and L being middle points of the sides, IJLK is a parallelogram (Problem 151).

$\triangle AIK = \frac{1}{4}\triangle ABC$,

$\triangle JDL = \frac{1}{4}\triangle DBC$,

and $\triangle AIK + \triangle JDL = \frac{1}{4}$ of area of quadrilateral, $= \frac{1}{2}$ area of parallelogram IKJL. But, drawing perpendiculars, $\triangle IOK + \triangle JOL = \frac{1}{2}IK \times NO + \frac{1}{2}JL \times OP = JL \times \frac{1}{2}NP = \frac{1}{2}$ area IJKL. So $\triangle AIK + \triangle JDL + \triangle IOK + \triangle JLO = AIOK + JDLO =$ area of parallelogram; also $CKOL + OIBJ =$ area of parallelogram; therefore $AIOK + JDLO = CLOK + IBJO$. But $AIOK = \frac{1}{2}IK \times (AM + NO) = \frac{1}{2}IK \times SO$ and $JDLO = \frac{1}{2}JL \times (RD + OP) = \frac{1}{2}JL \times OT$. Also, $IK = JL$ and $OT = OS$, therefore $AIOK = JDLO$. So $CKOL = OIBJ$. Hence, $AIOK = KOLC = OLDJ = OJBI$.

II.—Solution by C. C. ROBINS, Princeton, New Jersey; J. M. QUIROZ, Alamos, Sonora, Mexico; and WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Ohio.



Let ABCD be any quadrilateral. Draw the diagonals AC and BD, and bisect them at E and F respectively. Through each of these points draw a line parallel to the other diagonal. The point G where these lines intersect is the point required.

Proof.—The triangles AED and AEB are equal; also triangles CDE and CBE are equal, because on equal bases and between same parallels. Therefore AECD is one-half the figure; but $AECD = \triangle AEC + \triangle ADC = \triangle AGC + \triangle ADC$. In the same way DGBC, CGAB, BGDA are each one-half the figure. Subtracting from equation $AGCB = GBCD$ the common triangle BGC, we have $\triangle AGB = \triangle CGD$, and in similar manner $\triangle ADG = \triangle BGC$. Each of these triangles is bisected by the line drawn from G to the middle of its base, and adding halves of equals together we obtain $HGLC = KGLB = KGIA = GHDI$.

Solved also by Messrs. *Brown, Casey, Edmunds, Hart, Hawley, Matz, Place, Pollard, Rosell, Scheffer, Seitz, Shidy* and *Siverly*.

161.—Proposed by **Artemas Martin, M. A.**, Member of the London Mathematical Society, Erie, Erie County, Pa.
What rate per cent. of interest paid in advance is equivalent to r per cent. paid at the end of the year?

I.—Solution by the **Proposer.**

Let $x =$ the rate per cent. required expressed decimally.

If the interest is to be equivalent to the *lender* to r per cent. paid in advance it is the present worth of r discounted at r per cent.; $\therefore\ x = \dfrac{r}{1+r}$.

But if the interest is to be equivalent to the *borrower* to r per cent. paid in advance it is the present worth of r discounted at x per cent.; $\therefore\ x = \dfrac{r}{1+x}$, whence $x^2 + x = r$ and $x = \sqrt{(r+\frac{1}{4})} - \frac{1}{2}$.

II.—Solution by **Walter Siverly**, Oil City, Venango Co., Pa.; and **Sylvester Robins**, North Branch Depot, N. J.

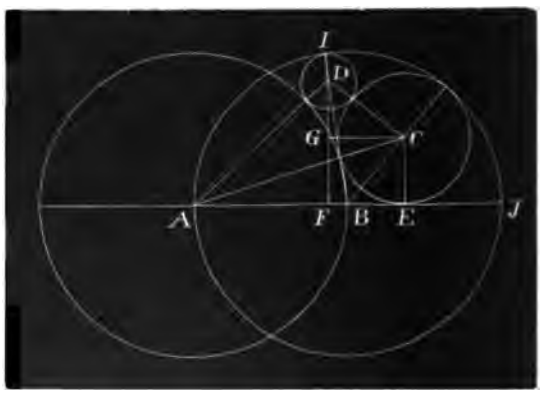
Let $x =$ the required rate per cent. in advance expressed decimally.
The interest on x is x^2; on x^2, x^3; on x^3, x^4; on x^4, x^5; etc.; etc., and

$$x + x^2 + x^3 + x^4 + x^5 + \text{etc.} = r, \text{ or } \frac{x}{1-x} = r; \text{ whence } x = \frac{r}{1+r}.$$

Solutions of the first case received from Messrs. *Bowser*, *Casey*, *Clark*, *Donovan*, *Drummond*, *Haynes*, *Pollard*, *Putnam*, *C. C. Robins*, *Shidy* and *Seitz*; and of the second case from Messrs. *Matz* and *Rosell*.

162.—Proposed by O. H. **Merrill**, Mannsville, Jefferson County, New York.
Two equal circles intersect, the center of each being on the circumference of the other. A circle is drawn touching that diameter of the right-hand circle which joins the centers of the given circles, and the circumferences of both circles, the right-hand one internally and the other externally; also a circle is drawn touching the one last drawn and the circumferences of both the given circles. Find the radii of these two circles.

Solution by **Artemas Martin, M. A.**, Member of the London Mathematical Society, Erie, Erie Co., Pennsylvania; and **Lucius Brown**, Hudson, Middlesex Co., Massachusetts.



1. Let r represent the radius of one of the equal circles, and x the radius of the required circle touching the diameter AJ.

Then $AC = r + x$,

$AE = \sqrt{[(r+x)^2 - x^2]} = \sqrt{(r^2+2rx)}$,

$BE = \sqrt{[(r-x)^2 - x^2]} = \sqrt{(r^2-2rx)}$;

$AB = AE - BE$, therefore

$r = \sqrt{(r^2+2rx)} - \sqrt{(r^2-2rx)} \ldots\ldots (1).$

Squaring we get

$2\sqrt{(r^2-4x^2)} = r$;

squaring again and reducing we find $x = \frac{1}{4}r\sqrt{3}$.

2. Draw DF perpendicular to the diameter AJ, and CG perpendicular to DF.

Let $y =$ radius of the second required circle, $AF = w$ and $DF = z$.

Then $BF = r - w$, $BE = \sqrt{[(r-\frac{1}{4}r\sqrt{3})^2 - (\frac{1}{4}r\sqrt{3})^2]} = \frac{1}{2}r(\sqrt{3}-1)$, $FE = BF + BE = \frac{1}{2}r(1+\sqrt{3}) - w$, $DG = z - \frac{1}{4}r\sqrt{3}$, $CD = y + \frac{1}{4}r\sqrt{3}$, $BD = r - y$, $AD = r + y$, and we have

$$(AF)^2 + (DF)^2 = (AD)^2,\quad (BF)^2 + (DF)^2 = (BD)^2,\quad (GC)^2 + (GD)^2 = (CD)^2;$$

or $\quad w^2 + z^2 = (r+y)^2 \ldots\ldots\ldots\ldots (2),\qquad (r-w)^2 + z^2 = (r-y)^2 \ldots\ldots\ldots\ldots (3),$

$$(\tfrac{1}{2}r + \tfrac{1}{2}r\sqrt{3} - w)^2 + (z - \tfrac{1}{4}r\sqrt{3})^2 = (y + \tfrac{1}{4}r\sqrt{3})^2 \ldots\ldots\ldots\ldots (4).$$

Subtracting (3) from (2), we get $\quad w = \frac{1}{2}r + 2y \ldots\ldots\ldots\ldots (5).$

Substituting in (2) and (4),

$$(\tfrac{1}{2}r + 2y)^2 + z^2 = (r+y)^2 \ldots\ldots (6),\qquad (\tfrac{1}{2}r\sqrt{3} - 2y)^2 + (z - \tfrac{1}{4}r\sqrt{3})^2 = (y + \tfrac{1}{4}r\sqrt{3})^2 \ldots\ldots (7).$$

Expanding (6) and (7), and reducing,

$$z^2 + 3y^2 = \tfrac{3}{4}r^2 \ldots\ldots\ldots\ldots (8),\qquad (z - \tfrac{1}{4}r\sqrt{3})^2 = \tfrac{5}{2}ry\sqrt{3} - 3y^2 - \tfrac{9}{16}r^2 \ldots\ldots\ldots\ldots (9).$$

Subtracting (8) from (9), we get $\quad z = r\sqrt{3} - 5y \ldots\ldots\ldots\ldots (10).$

Substituting in (8), $28y^2 - 10ry\sqrt{3} = -\frac{9}{4}r^2 \ldots\ldots (11)$; which quadratic gives $y = \frac{3}{28}r\sqrt{3}$.

Solved also by Messrs. *Hoover*, *Putnam*, *C. C. Robins*, *Scheffer* and *Siverly*. Prof. *Casey*, *J. W. Donovan* and *Edwin Place* each solved the first part.

163.—Proposed by THEODORE L. DELAND, Office of the Secretary of the Treasury, Washington, D. C.

In 1861 a 6-per-cent. 20-year coin bond of the U. S., interest payable semi-annually, sold on the market for $0.891 on the dollar; what, on this basis, would have been the market value of a 4-per-cent. 28-year coin bond of the U. S., interest payable quarterly?

I.—Solution by the PROPOSER.

Let $r=$ rate per annum paid on a bond, $n=$ the number of interest-payments in a year, $t=$ the number of years the bond runs, $v=$ the market value of the bond (or one dollar of it), and $R=$ the rate per annum the investor realizes (paid n times a year).

Then $\frac{r}{n}$ is the income of the investor during each of the n intervals, which he will receive nt times, receiving with the last the face of the bond, $1.00.

The payment preceding the last is at interest $\frac{1}{n}$ of a year at R per annum; and $\frac{r}{n}\left(1+\frac{R}{n}\right)$ is the value of that payment when the bond matures; and the sum of all the payments improved at the rate R $\left(\frac{1}{n}\text{-annually}\right)$ will equal v so improved.

Therefore we have the equation

$$1+\frac{r}{n}+\frac{r}{n}\left(1+\frac{R}{n}\right)+\frac{r}{n}\left(1+\frac{R}{n}\right)^2+\frac{r}{n}\left(1+\frac{R}{n}\right)^3+\ldots+\frac{r}{n}\left(1+\frac{R}{n}\right)^{tn-1}=v\left(1+\frac{R}{n}\right)^{tn}\ldots\ldots(1).$$

Summing, transposing, and reducing, we have

$$(Rv-r)\left(1+\frac{R}{n}\right)^{tn}=R-r\ldots\ldots\ldots\ldots(2);$$

or by logarithms,

$$tn\log\left(1+\frac{R}{n}\right)+\log(Rv-r)=\log(R-r)\ldots\ldots\ldots\ldots(3).$$

Taking the elements of the 6-per-cent. bond, we find from (3), by Double Position, $R=0.070226$.

For the constant rate per annum when paid n and n' times a year during any t years we have,

$$\left(1+\frac{R}{n}\right)^{tn}=\left(1+\frac{R'}{n'}\right)^{tn'}\ldots\ldots\ldots\ldots(4);$$

or after reduction

$$R'=n'\left(1+\frac{R}{n}\right)^{\frac{n}{n'}}-n'\ldots\ldots\ldots\ldots(5);$$

that is to say, equation (5) gives the rate the investor realizes when paid n' times a year instead of n times; therefore $R'=0.069620$, in the four-per-cent. bond.

In equation (1) let R become R' and n become n', or take the elements of the four-per-cent. bond, and we have

$$v'=\frac{R'-r'}{R'\left(1+\frac{R'}{n'}\right)^{t'n'}}+\frac{r'}{R'}=0.6361467;$$

or a $100-bond (at 4 per cent.) would sell for $63.61.

II.—Solution by M. H. DOOLITTLE, U. S. Coast Survey Office, Washington, D. C.

I assume that in the case of two bonds having the same rate of interest payable at the same intervals, and differing only in respect to the time at which the principal is payable, if either sells at par the other sells also at par. I know that bonds having a long time to run usually sell for more than similar bonds having a short time to run; but I suppose the difference to be fluctuating and uncertain; and unless data for its determination are given in such problems, it can not be taken into consideration.

I interpret the problem thus: At a certain rate per cent., not explicitly stated in the problem but determinable therefrom, upon a principal of $0.891, partial payments of 3 cents each are made at regular intervals of 6 months; and at the end of twenty years the amount due is $1.03. At the same rate per cent., upon the required principal, partial payments of 1 cent are made at regular intervals of 3 months; and at the end of twenty-eight years the amount due is $1.01.

Let $x=$ amount of $1.00 in three months at the above-mentioned rate per cent., and $y=$ the required principal.

Then $$0.891x^{80}-0.03x^{78}-0.03x^{76}-0.03x^{74}-\ldots\ldots-0.03x^2-1.03=0\ldots\ldots(1),$$

and $$yx^{112}-0.01x^{111}-0.01x^{110}-0.01x^{109}-\ldots\ldots-0.01x-1.01=0\ldots\ldots(2).$$

Multiplying (1) by x^2-1 and (2) by $x-1$,

$$0.891(x^{82}-x^{80})-0.03x^{80}-x^2+1.03=0\ldots\ldots(3),$$

$$y(x^{113}-x^{112})-0.01x^{112}-x+1.01=0\ldots\ldots(4).$$

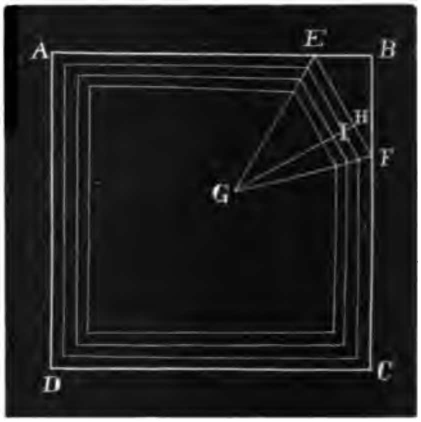
From (3), $x=1.017405$, and then from (4) $y=0.6361467$.

Mr. *Doolittle* solved this problem at the request of the *Proposer* who placed his solution in our hands for publication. Excellent solutions were also given by *F. W. Lantz*, Washington, D. C., and *Walter Siverly*, Oil City, Pa.

164.—Proposed by V. WEBSTER HEATH, Rodman, Jefferson County, New York.

A field 81 rods square has one of its corners clipped off by a line meeting the sides at 60 and 80 yards respectively from the corner. A man commences plowing around the field, turning a furrow one foot wide. How many "rounds" will he plow before the unplowed portion becomes a square?

Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.



Bisect the angles AEF and CFE and produce the bisecting lines to meet in G. Draw GH perpendicular to EF.

There will be as many "rounds" as the width IH of the furrow is contained times in GH.

Denoting EB by a, BF by b, EF by c, the width of the furrow by d, $\angle$BEF by α, $\angle$BFE by β, we easily obtain

$$GH=\frac{c}{\tan\frac{1}{2}\alpha+\tan\frac{1}{2}\beta}.\quad \text{But } \cos\alpha=\frac{a}{c},\ \cos\beta=\frac{b}{c},$$

$$\sin\alpha=\frac{b}{c},\ \sin\beta=\frac{a}{c}, \text{ and from the formula}$$

$$\tan\tfrac{1}{2}\varphi=\frac{1-\cos\varphi}{\sin\varphi} \text{ we have } \tan\tfrac{1}{2}\alpha=\frac{c-a}{b},\ \tan\tfrac{1}{2}\beta=\frac{c-b}{a}.$$

Substituting in the foregoing value of GH we get

$$GH=\frac{ab}{a+b-c}, \text{ and number of "rounds"}=\frac{ab}{d(a+b-c)}.$$

In our case $a=60$ yards, $b=80$ yards, $c=100$ yards, $d=1$ foot $=\frac{1}{3}$ of a yard, and the number of "rounds" is 360.

Good solutions also given by the *Proposer*, *Lucius Brown*, *J. A. Pollard*, *L. P. Shidy* and *Walter Siverly*.

165.—Proposed by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

A railroad company has $400000 of preferred stock and $300000 of common stock. The agreement is that the net income each year shall be applied to the payment of six per cent. on the preferred stock, and the balance shall be divided on the common stock. The net income is $36000 rental paid annually; a debt of $50000 is created payable in twenty years with annual interest at six per cent. (payable at the same time as the rent) in such manner that the annual interest is payable out of the current income, but the principal out of all the income after the debt becomes due until it is paid. It is agreed to create a sinking fund. What amount must be carried to the sinking fund annually, assuming that five per cent. compound interest may be earned to extinguish the debt when it becomes due, and how shall that amount apportioned on the two kinds of stock?

Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania; and the PROPOSER.

Let $x=$ the amount laid by in the sinking fund each year, $1.05=R$, $50000=a$. Then

$$x+Rx+R^2x+R^3x+\ldots\ldots+R^{19}x=a,\quad \frac{x(R^{20}-1)}{R-1}=a,\quad x=\frac{a(R-1)}{R^{20}-1}=\$1512.127.$$

The preferred stock under the original arrangement would pay $24000 at the end of 20 years and what the rental lacked of paying $24000 after paying off the debt at the end of the next year.

At the end of 20 years there is due, principal and interest, $53000, from which subtracting $36000 leaves $17000, plus interest = $18020 due at the end of the 21st year. $24000 + $18020 — $36000 = $6020 which the preferred stock would pay at that time, which is equivalent to $5679.244 paid at the end of the 20th year; this added to $24000 gives $29679.244, the portion of the $50000 paid by the preferred stock.

$50000 : $1512.127 :: $29679.244 : $897.575,

sum put in the sinking fund by the preferred stock each year, which subtracted from $1512.127 leaves $614.462 to be put in by the common stock.

166.—Proposed by Dr. S. HART WRIGHT, M. A., Ph. D., Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Required the variation v of the magnetic needle, in latitude $\lambda=42^\circ\,30'$, the declination of the North Star being $\delta=88^\circ\,40'$, and its magnetic bearing $b=8^\circ\,48'\,30''$ E. when at its greatest elongation east.

Solution by EDWARD A. BOWSER, Professor of Mathematics and Engineering, Rutgers College, New Brunswick, N. J.; and the PROPOSER.

Let $z=$ azimuth of North Star at its greatest eastern or western elongation.

Then, by works on Astronomy, we have $z=\sin^{-1}(\cos\delta\sec\lambda)=1^\circ\,48'\,30''$ which added to the magnetic bearing of the North Star when at its greatest elongation west, or subtracted from the bearing when at its greatest elongation east, gives us the variation of the magnetic needle.

$$\therefore\ v=b-z=(8^\circ\,48'\,30'')-(1^\circ\,48'\,30'')=7^\circ, \text{ the variation west of north.}$$

Solved also by *K. S. Putnam* and *Walter Siverly*.

167.—Proposed by B. F. BURLESON, Oneida Castle, Oneida County, New York.

Given $3(x^2+y^2)=2xy(x+y)$ and $xy(x^2-y^2)=13(x-y)$, to find accurate expressions for the values of x and y.

Solution by WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Ohio; the PROPOSER; E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri; and Dr. DAVID S. HART., M. A., Stonington, Connecticut.

Let $xy=p$, $x+y=s$, and the given equations readily become

$s^2-2p=\frac{1}{3}(26)\ldots\ldots.(1)$, and $ps=13\ldots\ldots.(2)$. These give $s^3-\frac{1}{3}(26)s=26\ldots\ldots.(3)$.

Put $s=\sqrt[3]{w}+\dfrac{26}{9\sqrt[3]{w}}$, and (3) becomes $w^2-26w=-\left(\dfrac{26}{9}\right)^3$; whence $w=\dfrac{26}{27}$ or $\dfrac{676}{27}$;

$\therefore\ s=\frac{1}{3}[\sqrt[3]{(676)}+\sqrt[3]{(26)}]$. Hence $p=\frac{1}{6}(3s^2-26)=\frac{1}{18}[26\sqrt[3]{(26)}+\sqrt[3]{(676)}-26]=xy$.

Having $x+y$ and xy we find

$$x=\tfrac{1}{2}[s+\sqrt{(s^2-4p)}]=\tfrac{1}{6}\{\sqrt[3]{(676)}+\sqrt[3]{(26)}+\sqrt{[4(26)-26\sqrt[3]{(26)}-\sqrt[3]{(676)}]}\},$$

$$y=\tfrac{1}{2}[s-\sqrt{(s^2-4p)}]=\tfrac{1}{6}\{\sqrt[3]{(676)}+\sqrt[3]{(26)}-\sqrt{[4(26)-26\sqrt[3]{(26)}-\sqrt[3]{(676)}]}\}.$$

Solutions received also from Messrs. *Casey*, *Edmunds*, *Matz*, *Putnam*, *Rosell* and *Sirerly*.

168.—Proposed by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.

The position of a ball on a circular billiard table is given. What path must the ball describe in order to pass through its original position after touching the cushion twice?

Solution by WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.

Let P be the given position of the ball, O the center of the circular table, A and C the points in which the ball strikes the cushion, and B the middle point of AC.

Let $OP=a$, $OA=OC=r$, $\angle OPA=\angle OPC=\varphi$; then

$$\angle PAO=\sin^{-1}\left(\frac{a\sin\varphi}{r}\right),=45^\circ-\tfrac{1}{2}\varphi$$

since the angle of incidence equals the angle of reflection;

$$\therefore\ \frac{a\sin\varphi}{r}=\sin(45^\circ-\tfrac{1}{2}\varphi)=\sqrt{[\tfrac{1}{2}(1-\sin\varphi)]},$$

whence $\sin^2\varphi+\dfrac{r^2\sin\varphi}{2a^2}=\dfrac{r^2}{2a^2}$, and $\sin\varphi=\dfrac{r\sqrt{(8a^2+r^2)}-r^2}{4a^2}$.

Good solutions given by the *Proposer* and Messrs. *Casey*, *Clark*, *Place*, *Putnam*, *C. C. Robins*, *Seitz*, *Shidy* and *Sirerly*.

169.—Proposed by H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C.

Required (1) the radius of the auger that will cut out one-mth part of the surface, and (2) the radius of the auger that will cut out one-nth of the volume of a sphere, radius r, the axis of the auger coinciding with a diameter of the sphere.

Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; and E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

1. Let $x=$ radius of the auger. The auger cuts out two segments, the surface of each of which is $2\pi rh$, h being $=r-\sqrt{(r^2-x^2)}$; $\therefore\ 4\pi r[r-\sqrt{(r^2-x^2)}]=\dfrac{4\pi r^2}{m}$, whence $x=r\left[1-\left(1-\dfrac{1}{m}\right)^2\right]^{\frac{1}{2}}$.

2. Let $y=$ radius of the auger. The volume removed is composed of two equal segments, radii of bases y and hight of each $=h'=r-\sqrt{(r^2-y^2)}$, and a cylinder radius y and altitude $2\sqrt{(r^2-y^2)}$. The volume of each segment is $\frac{1}{3}\pi h'^2(3r-h')$; the volume of the cylinder is $2\pi y^2\sqrt{(r^2-y^2)}$;

$$\therefore\ \tfrac{2}{3}\pi h'^2(3r-h')+2\pi y^2\sqrt{(r^2-y^2)}=\frac{\frac{4}{3}\pi r^3}{n}.$$

Substituting value of h' and dividing out π,

$$[2r^2-y^2-2r\sqrt{(r^2-y^2)}][2r+\sqrt{(r^2-y^2)}]+3y^2\sqrt{(r^2-y^2)}=\frac{2r^3}{n};$$

whence $\qquad (r^2-y^2)\sqrt{(r^2-y^2)}=r^3\left(1-\dfrac{1}{n}\right)$, and $y=r\left[1-\left(1-\dfrac{1}{n}\right)^{\frac{2}{3}}\right]^{\frac{1}{2}}$.

Solved also by the *Proposer* and Messrs. *Casey*, *Hart*, *Place*, *Putnam*, *C. C. Robins*, *Rosell*, *Shidy* and *Sirerly*.

170.—Proposed by K. S. PUTNAM, Rome, Oneida County, New York.

Traveling recently on a train moving 30 miles an hour and overhauling a freight moving 20 miles an hour, I endeavored to ascertain the length of the freight. One minute from the time I was opposite the rear of the freight a third train came between and put a stop to my investigation. At the next station we stopped, the door of the depot being directly opposite my seat. The freight here overhauled us, the front of the train being opposite me, when our train started. When we had attained the same rate of speed as the freight I was opposite the same point in the freight as when I closed my first observation.

It took our train $1\frac{1}{2}$ mintes to get under full headway, during which time its motion was uniformly accelerated. How long was the freight train, and how far was I from the depot when I passed the freight?

Solution by L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C.; E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri; LUCIUS BROWN, Hudson, Middlesex Co., Massachusetts; and C. C. ROBINS, Princeton, New Jersey.

Let P denote the passenger train, and F the freight train. Since P moves 30 miles an hour, it will move $\frac{1}{2}$ mile per minute; and since F moves 20 miles an hour, it will move $\frac{1}{3}$ mile per minute. Hence P gains on F $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ mile in 1 minute, when the observation was broken off by third train.

After stopping it took P 1 minute from the start to attain the same velocity as F; during this minute P's average velocity was 10 miles an hour, or $\frac{1}{6}$ of a mile per minute, while F traveling $\frac{1}{3}$ of a mile in the same time, gained $\frac{1}{3}-\frac{1}{6}=\frac{1}{6}$ of a mile on P. By the problem this brings us again to the point where our observations were broken off, so we have now seen the whole of F, and its length is $\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$ mile.

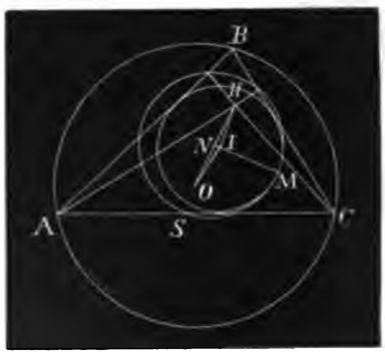
Since P took $1\frac{1}{2}$ minutes to attain full speed, its average speed was 15 miles an hour, and during this time it traveled $\frac{3}{8}$ of a mile. During the same time F traveled $\frac{1}{3}\times\frac{3}{2}=\frac{1}{2}$ mile, having gained $\frac{1}{2}-\frac{3}{8}=\frac{1}{8}$ of a mile on P. After attaining full speed P gains on F at the rate of $\frac{1}{6}$ of a mile per minute, hence to gain $\frac{1}{8}$ of a mile it will take $\frac{3}{4}$ of a minute, during which time it will travel $\frac{1}{2}\times\frac{3}{4}=\frac{3}{8}$ of a mile. Hence the entire distance from the depot when P passed F was $\frac{3}{8}+\frac{3}{8}=\frac{3}{4}$ of a mile.

Solved also by the *Proposer* and Messrs. *Donovan*, *Rosell* and *Siverly*.

171.—Proposed by MARCUS BAKER, U. S. Coast Survey Office, Washington, D. C.

In a plane triangle ABC the center of the circumscribed circle is O, the center of the inscribed circle is I and the intersection of the perpendiculars is H. Knowing the sides of the triangle OIH, determine the sides of the triangle ABC.

Solution by Professor W. P. CASEY, C. E., San Francisco, California.



Let OH $= a$, HI $= b$, IO $= c$, $r =$ radius of inscribed circle, $R =$ radius of circumscribing circle.

As the lines OH, HI and IO are given, therefore the points O, I and H are given in position. Bisect OH in N, then N is a given point, and is the center of the nine-point circle. Join N and I and produce NI to meet the circumference of the inscribed circle in M. As the nine-point circle and the inscribed circle are tangent circles, therefore M is the point of tangency and NM is the radius of the nine-point circle, $=\frac{1}{2}R$. Now NI becomes a known line being $=\frac{1}{2}[2(\mathrm{HI})^2+2(\mathrm{OI})^2-4(\mathrm{NH})^2]^{\frac{1}{2}}$

$$=\tfrac{1}{2}(2b^2+2c^2-a^2)^{\frac{1}{2}}=p=\mathrm{NM}-\mathrm{IM}=\tfrac{1}{2}R-r,$$

and $(\mathrm{OI})^2=R^2-2Rr$; whence R and r become known lines, and therefore the three circles are given in position and magnitude. It only remains to find a point C in the circumference of ABC so that the tangent CA to circle I may be bisected by the circle N in the point S, which is easily done. But the angles of the triangle ABC may be found from the following well-known properties, viz;

$$\frac{r}{R}=\cos\mathrm{A}+\cos\mathrm{B}+\cos\mathrm{C}-1,\ \text{or}\ \cos\mathrm{A}+\cos\mathrm{B}+\cos\mathrm{C}=1+\frac{r}{R}\ \ldots\ldots\ldots\ldots(1);$$

$$(\mathrm{OH})^2=2R^2(\tfrac{3}{2}+\cos2\mathrm{A}+\cos2\mathrm{B}+\cos2\mathrm{C}),\ \text{from which we get}$$

$$\cos^2\mathrm{A}+\cos^2\mathrm{B}+\cos^2\mathrm{C}=\frac{a^2+3R^2}{4R^2}\ldots\ldots(2);\quad \cos\mathrm{A}\cos\mathrm{B}\cos\mathrm{C}=\frac{R^2-(\mathrm{OH})^2}{8R^2}=\frac{R^2-a^2}{8R^2}\ldots\ldots(3).$$

From (1) and (2) we have $\cos\mathrm{A}\cos\mathrm{B}+\cos\mathrm{A}\cos\mathrm{C}+\cos\mathrm{B}\cos\mathrm{C}=\dfrac{(R+r)^2}{2R^2}-\dfrac{a^2+3R^2}{8R^2}\ldots\ldots\ldots(4),$

From (1), (4) and (3) we have immediately by the Theory of Equations the cubic equation

$$\cos^3\mathrm{A}-\left(1+\frac{r}{R}\right)\cos^2\mathrm{A}+\left(\frac{(R+r)^2}{2R^2}-\frac{a^2+3R^2}{8R^2}\right)\cos\mathrm{A}=\frac{R^2-a^2}{8R^2},$$

the three roots of which are the values of cos A, cos B and cos C.

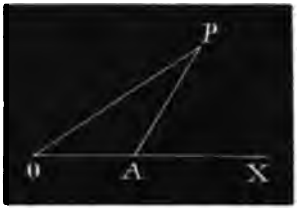
The sides of the triangle ABC are AB $=2R\sin\mathrm{C}$, BC $=2R\sin\mathrm{A}$ and AC $=2R\sin\mathrm{B}$.

Excellent solutions received from *Lucius Brown*, Professor *Scheffer* and Professor *Seitz*.

172.—Proposed by W. W. JOHNSON, M. L. M. S., Professor of Mathematics, St. John's College, Annapolis, Maryland.

The base of a triangle is fixed and the difference between the vertical angle and one of the angles at the base is constant. Find the locus of the vertex. Discuss the curve and consider the cases in which the constant difference is 0 and 90°.

Solution by the PROPOSER.



Let OAP be the triangle; take O as the origin, and the base OA as the axis of x.

Let POA $= \theta$, PAX $= \varphi$, the constant difference POA — OPA $= \alpha$, and OA $= a$. The vertical angle is then $\varphi - \theta$ and the condition of the problem gives $2\theta - \varphi = \alpha$ .... (1).

Now $\tan\theta = \dfrac{y}{x}$, and $\tan\varphi = \dfrac{y}{x-a}$ .... (2),

and (1) gives $\tan 2\theta = \tan(\varphi + \alpha)$, threfore $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{\tan\alpha + \tan\varphi}{1-\tan\alpha\tan\varphi}$,

or substituting from (2) $\dfrac{2xy}{x^2-y^2} = \dfrac{(x-a)\tan\alpha + y}{x-a-y\tan\alpha}$, reducing to

$$(x^2+y^2)(y-x\tan\alpha) + a(x^2-y^2)\tan\alpha - 2axy = 0 \quad\ldots\ldots\ldots (3),$$

the equation of the locus required.

Writing the equation in the form

$$y - x\tan\alpha = \frac{2axy - a(x^2-y^2)\tan\alpha}{x^2+y^2},$$

we have when $y = x\tan\alpha = \infty$,

$$y - x\tan\alpha = a\tan\alpha \quad\ldots\ldots\ldots (4),$$

which is therefore the equation of the asymptote.

Equating to zero the terms of the lowest degree in (3), we have

$$y = x(-\cot\alpha \pm \operatorname{cosec}\alpha),$$

the equation of two tangents at the origin, whose inclinations to the axis of x are therefore $\frac{1}{2}\alpha$ and $90° + \frac{1}{2}\alpha$.

These conclusions can also be derived geometrically thus; when P is at infinity $\theta = \varphi$, hence from (1) $\theta = \varphi = \alpha$ the inclination of the asymptote. Now the line AP rotates with twice the angular velocity of OP, since from (1) $d\varphi = 2d\theta$; hence when OP and AP are parallel the distance of P from AP is double its distance from OP; therefore the asymptote cuts OX produced at a distance a on the left of the origin, which agrees with (4).

When $\varphi = 0$, π, 2π, &c., P comes into coincidence with O, and OP becomes a tangent to the curve. Now (1) gives for these values $\theta = \frac{1}{2}(\alpha + k\pi)$, (where k is an integer,) which gives the direction of the two tangents as before.

When $\alpha = 90°$ we have $x(x^2+y^2) - a(x^2-y^2) = 0$ .... (5),

of which the asymptote is $x = -a$, and the two tangents at the origin $y = \pm x$.

The general case (3) is known by French writers as the *oblique strophoid*, and the case (5), or the *right strophoid*, has been discussed by Dr. Booth under the name of the *logocyclic curve*.—(*Booth's New Geometrical Methods*, Vol. I, p. 295. See also *Rice and Johnson's Differential Calculus*, p. 290.)

When $\alpha = 0$ (3) breaks up into $y = 0$ and $x^2 + y^2 - 2ax = 0$; that is, the axis of x and the circle whose center is A and radius a, as is geometrically evident should be the case.

Good solutions by Messrs. *Bowser, Casey, Edmunds, Hoover, Ludwig, Rosell, Scheffer, Seitz* and *Siverly*.

173.—Proposed by Dr. S. H. WRIGHT, M. A., Ph. D., late Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Required the variation v of the magnetic needle, in latitude $\lambda = 42°\ 30'$, the declination of the sun being $\delta = 20°$ N., and the magnetic bearing of its upper limb when rising on a horizon elevated $h = 1°$ is $b = 69°\ 21'\ 40''$. Radius of sun $= r = 16'$, and refraction $= p = 35'$.

Solution by the PROPOSER.

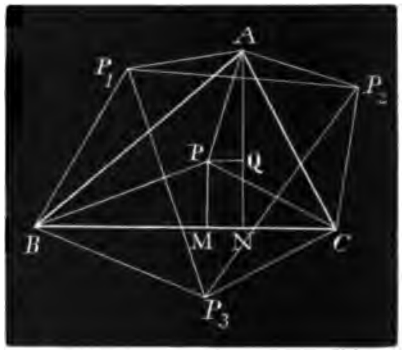
The zenith distance of the sun's center $= 90° - h + p + r = 89°\ 51' = z$. Put $\frac{1}{2}(z + \lambda + \delta) = s$. The true azimuth is $2\cos^{-1}\sqrt{[\cos(s-z)\sin(s-\lambda)\sec\lambda\operatorname{cosec}z]} = 62°\ 30'\ 58'' = a$. The magnetic azimuth is b, and $> a$, therefore $b - a = 6°\ 50'\ 42''$, the variation west of north.

Solved also by *K. S. Putnam*, Professor *E. B. Seitz* and *Walter Siverly*.

174.—Proposed by the late BENJAMIN PEIRCE, LL. D., F. R. S., Professor of Mathematics, Harvard University, and Consulting Geometer to the U. S. Coast Survey, Cambridge, Massachusetts.

To find by quadratic equations a triangle of which the angles are given and the distances of the vertices from a given point in the plane of the triangle.

I.—Solution by WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.



Let A, B and C be the angles and a, b, c the given distances from the vertices to the point P in the triangle.

With vertical angles 2A, 2B and 2C, and including sides a, b and c, construct the isosceles triangles AP_1P_2, BP_1P_3, CP_2P_3, and arrange them so that their bases shall form the triangle $P_1P_2P_3$, their vertices lying without if the given angles are all acute; but if one of them is obtuse the corresponding vertex will lie within.

Join A, B and C, and ABC is the required triangle; $AP = a$, $BP = b$, $CP = c$.

We easily find $\quad P_1P_2 = 2a\sin A, \quad P_1P_3 = 2b\sin B$

and $\quad P_2P_3 = 2c\sin C$;

$$\angle P_2P_1P_3 = \cos^{-1}\left(\frac{a^2\sin^2 A + b^2\sin^2 B - c^2\sin^2 C}{2ab\sin A\sin B}\right), \quad \angle AP_1B = C + \cos^{-1}\left(\frac{a^2\sin^2 A + b^2\sin^2 B - c^2\sin^2 C}{2ab\sin A\sin B}\right).$$

Then from the triangle AP_1B we have

$$AB = \left\{a^2 + b^2 - 2ab\cos\left[C + \cos^{-1}\left(\frac{a^2\sin^2 A + b^2\sin^2 B - c^2\sin^2 C}{2ab\sin A\sin B}\right)\right]\right\}^{\frac{1}{2}},$$

$$= \{a^2 + b^2 - \operatorname{cosec} A\operatorname{cosec} B\,\{\cos C(a^2\sin^2 A + b^2\sin^2 B - c^2\sin^2 C) - \sin C\sqrt{[(2ab\sin A\sin B)^2 - (a^2\sin^2 A + b^2\sin^2 B - c^2\sin^2 C)^2]}\,\}\}^{\frac{1}{2}}.$$

Similarly, we find

$$AC = \left\{a^2 + c^2 - 2ac\cos\left[B + \cos^{-1}\left(\frac{a^2\sin^2 A + c^2\sin^2 C - b^2\sin^2 B}{2ac\sin A\sin C}\right)\right]\right\}^{\frac{1}{2}},$$

$$= \{a^2 + c^2 - \operatorname{cosec} A\operatorname{cosec} C\,\{\cos B(a^2\sin^2 A + c^2\sin^2 C - b^2\sin^2 B) - \sin B\sqrt{[(2ac\sin A\sin C)^2 - (a^2\sin^2 A + c^2\sin^2 C - b^2\sin^2 B)^2]}\,\}\}^{\frac{1}{2}}.$$

$$BC = \left\{b^2 + c^2 - 2bc\cos\left[A + \cos^{-1}\left(\frac{b^2\sin^2 B + c^2\sin^2 C - a^2\sin^2 A}{2bc\sin B\sin C}\right)\right]\right\}^{\frac{1}{2}},$$

$$= \{b^2 + c^2 - \operatorname{cosec} B\operatorname{cosec} C\,\{\cos A(b^2\sin^2 B + c^2\sin^2 C - a^2\sin^2 A) - \sin A\sqrt{[(2bc\sin B\sin C)^2 - (b^2\sin^2 B + c^2\sin^2 C - a^2\sin^2 A)^2]}\,\}\}^{\frac{1}{2}}.$$

II.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Let ABC be the required triangle, P the given point, PM, AN perpendiculars on BC, and PQ perpendicular to AN; $BP = a$, $CP = b$, $AP = c$, $BM = x$, $PM = y$, $BC = z$. Then will $BN = z\sin C\cos B\operatorname{cosec} A = mz$, $AN = z\sin B\sin C\operatorname{cosec} A = nz$, and we have

$$x^2 + y^2 = a^2 \ldots\ldots (1), \quad (z - x)^2 + y^2 = b^2 \ldots\ldots (2), \quad (mz - x)^2 + (nz - y)^2 = c^2 \ldots\ldots (3).$$

Substituting a^2 for $x^2 + y^2$ in (2) and (3),

$$z^2 - 2xz = b^2 - a^2 \ldots\ldots\ldots (4), \qquad (m^2 + n^2)z^2 - 2mxz - 2nyz = c^2 - a^2 \ldots\ldots\ldots (5).$$

Eliminating xz from (4) and (5),

$$(m^2 + n^2 - m)z^2 - 2nyz = c^2 - a^2 - m(b^2 - a^2), \quad \text{or} \quad y = \frac{(m^2 + n^2 - m)z^2 + m(b^2 - a^2) + a^2 - c^2}{2nz}.$$

From (4), $$x = \frac{z^2 + a^2 - b^2}{2z}.$$

Substituting these values of x and y in (1),

$$[(m^2 + n^2 - m)^2 + n^2]z^4 + [2(m^2 + n^2 - m)(mb^2 - ma^2 + a^2 - c^2) - 2n^2(a^2 + b^2)]z^2$$
$$= -(mb^2 - ma^2 + a^2 - c^2)^2 - n^2(a^2 - b^2)^2,$$

a quadratic for determining z.

For a geometrical construction, see *Simpson's Algebra*, p. 367.

Solutions received also from Messrs. *Casey*, *Eastwood*, *Rosell*, *Schæffer* and *Seitz*.

List of Contributors to the Junior Department.

WALTER SIVERLY, Oil City. Venango Co., Pa., solved all the problems but 171. C. A. O ROSELL, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa., all but 162, 163, 164, 165, 166, 171 and 173. E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo., all but 148, 152, 153, 159, 162, 163, 164, 165 and 166. K. S. PUTNAM, Rome, N. Y., all but 148, 149, 150, 153, 163, 164, 165, 171, 172 and 174. Prof. W. P. CASEY, C. E., San Francisco, Cal., all but 150, 153, 157, 158, 159, 2d part 162, 163, 164, 165, 170 and 173. J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa., all but 148, 149, 150, 156, 159, 161, 163, 165, 166, 167, 170 and 173. C. C. ROBINS, Princeton, N. J., all but 152, 153, 156, 157, 163, 165, 166, 167, 171, 172, 173 and 174. L. P. SHIDY, U. S. Coast Survey Office, Washington, D. C., all but 150, 153, 156, 157, 162, 163, 165, 166, 167, 171, 172, 173 and 174. WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, O., all but 150, 153, 155, 157, 161, 163, 164, 165, 166, 169, 171 and 173. Dr. DAVID S. HART, M. A., Stonington, Conn., solved 148, 149, 151, 154, 155, 157, 158, 159, 160, 167 and 169. F. P. MATZ, M. A., late Professor of Mathematics, Military and Scientific School, King's Mountain, N. C., 149, 150, 151, 152, 154, 155, 157, 158, 160, 161 and 167. SYLVESTER ROBINS, North Branch Depot, N. J., 148, 149, 150, 151, 152, 153, 155, 157, 158, 159 and 161. JOHN I. CLARK, Moran, Ind., 148, 149, 150, 151, 155, 156, 159, 160, 161 and 168. J. A. POLLARD, Goshen, N. Y., 148, 149, 150, 151, 152, 155, 156, 159, 160, 161 and 164. Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Me., 148, 151, 152, 154, 155, 157, 159, 161 and 165. J. W. DONOVAN, Ansonia, O., 148, 149, 150, 151, 155, 156, 159, 161 and 1st part of 162. EDWIN PLACE, Cincinnatus, N. Y., 148, 151, 155, 158, 1st part of 162, 168 and 169. LUCIUS BROWN, Hudson, Mass., 148, 152, 160, 162, 164, 170 and 171. H. T. J. LUDWIG, Professor of Mathematics, North Carolina College, Mount Pleasant, N. C., 148, 149, 151, 154, 155, 169 and 172. E. J. EDMUNDS, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana, 150, 155, 158, 160, 167 and 172. JOSEPH TURNBULL, East Liverpool, O., 148, 151, 152, 154, 155 and 158. Prof. H. S. BANKS, Instructor in English and Classical Literature, Newburg, N. Y., 148, 149, 151, 153, 154 and 155. A. E. HAYNES, M. Ph., Professor of Mathematics and Physics, Hillsdale College, Hillsdale, Mich., 149, 150, 155, 158 and 161. O. D. OATHOUT, Luana, Iowa, 148, 149, 151, 154 and 155. J. M. QUIROZ, Alamos, Sonora, Mexico, 148, 151, 155, 158 and 160. GEORGE HAWLEY, San Francisco, Cal., 149, 151, 155, 158 and 160. WALTER S. NICHOLS, Editor *Insurance Monitor*, New York, N. Y., 148, 149, 151, 152 and 159. P. F. MANGE, Alamos, Sonora, Mexico, 151, 155 and 158. EDWARD A. BOWSER, Professor of Mathematics and Engineering, Rutgers College, New Brunswick, N. J., 161, 166 and 172. THEODORE L. DELAND, U. S. Treasury Department, Washington, D. C., 156 and 163. J. M. TAYLOR, Milton, Oregon, 149 and 159. Dr. S. H. WRIGHT, M. A., Ph. D., Penn Yan, N. Y., 166 and 173. M. H. DOOLITTLE, U. S. Coast Survey Office, Washington, D. C., 163. GEORGE EASTWOOD, Saxonville, Massachusetts, 174. W. W. JOHNSON, Member of the London Mathematical Society, Professor of Mathematics, St. John's College, Annapolis, Md., 172. F. W. LANTZ, Washington, D. C., 168. FRANK T. FREELAND, Instructor in Mechanics, University of Pennsylvania, 159. B. F. BURLESON, Oneida Castle, N. Y., 167. V. W. HEATH, Rodman, N. Y., 164. E. A. SQUIER, Le Grand, Iowa, 159. CHARLES GILPIN, Jr., Philadelphia, Pa., 149. THOMAS BAGOT, County Superintendent, New Marion, Ind., 149.

PROBLEMS.

254.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society Erie, Erie County, Pa.

It is required to compute the number indicated by 4^{4^4}.

255.—Proposed by REUBEN KNECHT, Photographer, Easton, Northampton County, Pennsylvania.

A and B are equal partners in a house and share equal profits. A receives $986.67 for rent and pays for repairs $854.12, also pays B $112.50. B pays $72.95 for repairs. Now does A owe B or B owe A, and how much?

256.—Proposed by JOHN I. CLARK, Moran, Clinton County, Indiana.

Mr. B offered a certain railroad company 60 rods square of ground, on condition that they would make a station at a certain crossing. The company claimed 78 feet in width for the road, and agree to accept of Mr. B's offer, if he will give that amount of ground exclusive of the 78 feet. The road is to enter said plat 15 rods *west* of the south-east corner, and leave it 20 rods *east* of the north-west corner. If Mr. B agrees to this, how many rods square must he donate?

257.—From MCLELLAN'S TEACHER'S HAND-BOOK OF ALGEBRA, p. 225, Ex. 8.

A cask, A, contains m gallons of wine and n gallons of water; and another cask, B, contains p gallons of wine and q gallons of water. How many gallons must be drawn from each cask so as to produce by their mixture b gallons of wine and c gallons of water?

258.—Proposed by D. L. WRIGHT, Mallet Creek, Medina County, Ohio.

What per cent. of income do U. S. $4\frac{1}{2}$-per-cent bonds at 103 yield in currency when gold is 105?

259.—Proposed by WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.

Show that in any plane triangle the angle included between the perpendicular from the vertical angle and its bisector equals half the difference of the angles at the base.

260.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Four men, A, B, C and D, start at the same time from the same point in the circumference of a circular island 200 miles in circumference, and travel the same way around it; A going 16 miles a day; B, 23 miles a day; C, 37 miles a day; and D, 44 miles. In how many days will they all be together again, and where? A complete general solution is desired.

261.—Proposed by A. E. Haynes, M. Ph., Professor of Mathematics and Physics, Hillsdale College, Hillsdale, Mich.
If a, b and c be unequal and positive quantities, prove that $a^3+b^3+c^3 > 3abc$.

262.—Proposed by L. P. Shidy, U. S. Coast Survey Office, Washington, D. C.
A man deposits D dollars every year in a Bank, which allows him compound interest at the rate of r per cent. per annum. What sum will the Bank owe him at the end of n years?

263.—Proposed by Sylvester Robins, North Branch Depot, Somerset County, New Jersey.
How many acres of prairie sod can a farmer break up in all the years that he can turn over a rectangular piece of different size and integral dimensions every summer, surrounded by an outside furrow 100 rods in length?

264.—Proposed by Joseph H. Kershner, Professor of Mathematics, Mercersburg College, Mercersburg, Pa.
If $A \pm B = 90^\circ$, in a plane triangle, sides a, b, c,

$$2c^{\pm 2} = (a+b)^{\pm 2} + (a-b)^{\pm 2},$$

the upper and lower signs being taken together.

265.—Proposed by J. F. Bunn, Judge of Probate Court, Tiffin, Seneca County, Ohio.
Determine the relation between the product of the sides of a given triangle and another triangle formed by joining the centers of the escribed circles of the given triangle.

266.—Proposed by W. F. L. Sanders, New Albany, Floyd County, Indiana.
Two boys, John and James, ran a foot-race of 200 yards. At first, John gave James 2 seconds and 8 yards the start, and then beat him 2 seconds. The next time, he gave James 16 yards and 5 seconds the start, and was beaten 20 yards. In what time can each run the distance?

267.—Proposed by Theodore L. DeLand, Office of the Secretary of the Treasury, Washington, D. C.
U. S. 4-per-cent. bonds with 26 years to run, interest payable quarterly, are worth on the market 112. Consider this the measure of the National credit. The Secretary of the Treasury wishes to place on the market a new loan to refund maturing bonds—the 6's; the loan to run 40 years, interest payable tri-annually. Required the rate the new bonds must draw in order to sell at par.

268.—Proposed by Artemas Martin, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.
The distance between the centers of two circles, radii R and r, is a; find the radius of a circle touching them and their common tangent.

269.—Proposed by P. F. Mange, Alamos, Sonora, Mexico.
Let a line be drawn from the center of a circle to any point of any chord; then show that the square of this line, plus the rectangle of the segments of the chord, is equal to the square of the radius.

270.—Proposed by E. J. Edmunds, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana.
Circumscribe a circle about a triangle ABC; draw AD, BE perpendicular to BC and AC, and intersecting in P; produce AD to meet the circle in F, and then show that DP = DF.

271.—Proposed by Marcus Baker, U. S. Coast and Geodetic Survey Office, Washington, D. C.
From any point P in the periphery of an ellipse lines are drawn to the foci F and F′. Required the locus of the center of the circle inscribed in the triangle FPF′.

272.—Proposed by Benjamin Headley, Dillsboro, Dearborn County, Indiana.
Divide a board 10 feet long and 2 feet wide into four pieces, so that they can be put together to form a square.

273.—Proposed by J. M. Quiroz, Alamos, Sonora, Mexico.
The area of any triangle is equal to the radius of the circumscribed circle multiplied by half the perimeter of the triangle formed by joining the feet of the perpendiculars of the given triangle.

274.—From Dr. Morrison's Trigonometry, p. 221, Ex. 27.
If the middle points of the sides of a triangle be joined with the opposite angles, and R_1, R_2, R_3, ... R_6 be the radii of the circles described about the six triangles so formed, and r_1, r_2, r_3, ... r_6 the radii of the circles inscribed in the same, prove that

$$(1)\quad R_1R_3R_5 = R_2R_4R_6 \quad\text{and}\quad (2)\quad \frac{1}{r_1}+\frac{1}{r_3}+\frac{1}{r_5} = \frac{1}{r_2}+\frac{1}{r_4}+\frac{1}{r_6}.$$

275.—Proposed by William Woolsey Johnson, Member of the London Mathematical Society, Professor of Mathematics, St. John's College, Annapolis, Maryland.
If AB is the transverse axis of a hyperbola whose asymptotes make angles of 60 degrees with its axis, and D a point so taken that BD = ½AB; then if P be taken on the branch passing through B the angle PDA is double the angle PAD, and if P be on the other branch, the supplement of PDA is double the supplement of PAD.

Solutions of these problems should be received by September 1, 1881.

SENIOR DEPARTMENT.

Solutions of the Problems Proposed in No. 4.

175.—Proposed by W. L. HARVEY, Maxfield, Penobscot County, Maine.

A man buys a farm for \$4000 and agrees to pay it in 4 equal annual installments, interest at 5 per cent. per annum, compounded every instant. Required the annual payment.

I.—Solution by CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $a=\$4000$, $r=0.05$ and $x=$ annual payment. If the interest is compounded n times a year we have the present value of 1st installment $=x(1+n^{-1}r)^{-n}$; of the second, $=x(1+n^{-1}r)^{-2n}$; of the third, $=x(1+n^{-1}r)^{-3n}$; and of the fourth, $=x(1+n^{-1}r)^{-4n}$.

Summing we have, since the sum of the present worths of the payments must be equal to the debt,

$$a=\frac{x[1-(1+n^{-1}r)^{-4n}]}{(1+n^{-1}r)^n-1}; \qquad \therefore x=\frac{a[(1+n^{-1}r)^n-1]}{1-(1+n^{-1}r)^{-4n}}.$$

Since $(1+n^{-1}r)^n=1+\frac{n}{1}.n^{-1}r+\frac{n}{1}.\frac{n-1}{2}.n^{-2}r^2+\frac{n}{1}.\frac{n-1}{2}.\frac{n-2}{3}.n^{-3}r^3+\ldots\ldots$

$$=1+r+\frac{r^2}{1.2}+\frac{r^3}{1.2.3}+\ldots\ldots \text{ (if } n=\infty\text{)}, =\varepsilon^r\ (\varepsilon=\text{Naperian base}),$$

we have if interest is compounded every instant,

$$x=\frac{a(\varepsilon^r-1)}{1-\varepsilon^{-4r}}=\$4000\times\frac{\varepsilon^{0.05}-1}{1-\varepsilon^{-0.2}}=\$1131.38.$$

II.—Solution by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri.

Let $a=\$4000$, $p=$ the annual payment, $r=5$ per cent., and $n=4$, the number of payments.

To find the amount of a for n years, let $x=$ the amount at the end of t years; then we have

$$dx=rxdt,$$

whence by integration, observing that when $t=0$, $x=a$, we find

$$\log\left(\frac{x}{a}\right)=rt.$$

When $t=n$, we have $\log\left(\frac{x}{a}\right)=nr$, or $x=a\varepsilon^{nr}$, the amount of a for n years.

The amount of the first payment for $(n-1)$ years is $p\varepsilon^{(n-1)r}$, that of the second payment for $(n-2)$ years is $p\varepsilon^{(n-2)r}$, and so on. But the sum of the amounts of the payments is equal to the amount of the principal; hence we have

$$p(\varepsilon^{(n-1)r}+\varepsilon^{(n-2)r}+\varepsilon^{(n-3)r}+\ldots\ldots+1)=a\varepsilon^{nr},$$

whence by summing the series, and solving for p, we find

$$p=\frac{a\varepsilon^{nr}(\varepsilon^r-1)}{\varepsilon^{nr}-1}=\frac{a(\varepsilon^r-1)}{1-\varepsilon^{-nr}}=\$1131.37+.$$

Solved also by the *Proposer* and Messrs. *Eastwood*, *Hoover*, *Pollard*, *Rosell*, *Schaffer*, *Siverly* and *Wood*.

176.—Proposed by E. J EDMUNDS, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana.

The sides of an inscribed quadrilateral are the roots of a given equation of the fourth degree whose roots are all real and positive. Find the area of the quadrilateral, and the radius of the circumscribing circle, in functions of the coefficients of the equation.

Solution by WALTER SIVERLY, Oil City, Venango Co., Pa.; Prof. W. P. CASEY, C. E., San Francisco, Cal.; E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.; J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.; and the PROPOSER.

Let a, b, c, d be the sides of the quadrilateral, A its area, R the radius of the circumscribing circle and $x^4-mx^3+nx^2-px+q=0$ the equation.

Then $\quad m=a+b+c+d=2s, \qquad n=ab+ac+ad+bc+bd+cd,$

$$p=abc+abd+acd+bcd, \qquad q=abcd.$$

Also, $A=\surd[(s-a)(s-b)(s-c)(s-d)]=\surd(s^4-ms^3+ns^2-ps+q)$. But $s=\frac{1}{2}m$, therefore

$$A=\tfrac{1}{4}\surd[\tfrac{1}{2}(8q-4mp+2m^2n-m^4)].$$

Again, $$R=\tfrac{1}{4}\sqrt{\left(\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}\right)}.$$

But $\quad (ab+cd)(ac+bd)(ad+bc) = a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2+abcd(a^2+b^2+c^2+d^2),$

and $\quad a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2 = p^2-2nq, \qquad abcd(a^2+b^2+c^2+d^2) = q(m^2-2n);$

$$\therefore\; R=\tfrac{1}{2}\sqrt{\left(\frac{p^2+m^2q-4nq}{8q-4mp+2m^2n-m^4}\right)}.$$

Solved also by Messrs. *Eastwood*, *Hoover*, *Kummell*, *Rosell* and *Wood*.

177.—Proposed by JOSEPH H. KERSHNER, Professor of Mathematics, Mercersburg College, Mercersburg, Franklin Co., Pa.

Find three square numbers the ratio of whose sum and product shall be a square.

I.—Solution by GEORGE EASTWOOD, Saxonville, Middlesex County, Massachusetts.

Designate the required numbers by x^2, y^2, z^2; then, per question, $\dfrac{x^2+y^2+z^2}{x^2y^2z^2}$ is to be a square. The denominator is a square already, so that it only remains to make the numerator a square.

Put $y^2=p^2x^2$, $z^2=q^2x^2$; then $x^2+p^2x^2+q^2x^2=x^2(1+p^2+q^2)=x^2(1+p^2+\frac{1}{4}p^4)=x^2(1+\frac{1}{2}p^2)^2$, when $q=\frac{1}{2}p^2$.

Ex.—Take $p=4$, and $x=1$; then 1, 16 and 64 are the squares that will satisfy the conditions of the question. But an infinite number of other squares may be found.

II.—Solution by W. E. HEAL, Marion, Grant County, Indiana.

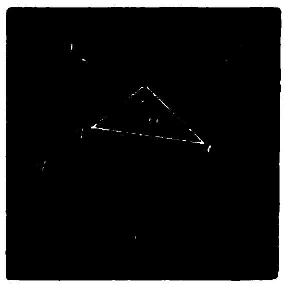
All we have to do is to find three square numbers whose sum shall be a square. This is satisfied by $(x^2+y^2+z^2)^2$, $4x^2z^2$, $4y^2z^2$, where x, y, z are any rational numbers. Taking $x=3$, $y=2$, $z=1$ we get 144, 36 and 16 for one set of the required numbers. $x=4$, $y=2$, $z=3$ gives 121, 576, 144.

Good solutions received from Messrs. *Drummond*, *Hart*, *Hoover*, *Pollard*, *C. C. Robins*, *Sylvester Robins*, *Scheffer*, *Shidy* and *Siverly*.

178.—Proposed by Professor W. P. CASEY, C. E., San Francisco, California.

Given the three lines joining the remote angles of the equilateral triangles described on the sides of a triangle, to construct the triangle and find its sides.

I.—Solution by E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri; and CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey Office, Washington, D. C.



Let DEF be the triangle formed by the three given lines. It is known that the three lines joining the vertices of the equilateral triangles to the opposite vertices of the original triangle are equal, that they meet in one point, and intersect at angles of 120°. Hence to determine their common point construct the isosceles triangles DHF and EKF, making the angles DHF, EKF each equal to 120°; with H and K as centers and radii equal to HD and KE describe arcs intersecting at O; then $\angle$DOF = $\angle$EOF = $\angle$DOE = 120°; therefore O is the common point of the three lines above named.

Again, it is known that the sum of the distances from the intersection of these three lines to the vertices of the original triangle is equal to one of the lines; hence the sum of the lines DO, EO and FO is twice one of the three equal lines.

Produce DO, EO, FO, making DC = BE = AF = $\frac{1}{2}$(DO + EO + FO);

join AB, AC, BC, AD, AE, BD, BF, CE, CF; then ABC is the original triangle.

To compute the sides of ABC, let EF $= a$, FD $= b$, DE $= c$, $a+b+c=2s$, area DEF $= u$, and $\angle$DFE $=$ F. Then we have FK $= \frac{1}{2}a\sqrt{3}$, FH $= \frac{1}{2}b\sqrt{3}$, $u = \frac{1}{2}ab\sin F = \sqrt{[s(s-a)(s-b)(s-c)]}$,

$$\text{OF}\times\text{HK} = 2\text{FK}\times\text{FH}\times\sin(60^\circ+\text{F}), \quad \text{or} \quad \text{OF} = \frac{2\text{FK.FH.}\sin(60^\circ+\text{F})}{\sqrt{[(\text{FK})^2+(\text{FH})^2-2\text{FK.FH.}\cos(60^\circ+\text{F})]}},$$

$$= \frac{2ab\sin(60^\circ+\text{F})}{\sqrt{[3a^2+3b^2-6ab\cos(60^\circ+\text{F})]}}, = \frac{(a^2+b^2-c^2)\sqrt{3}+4u}{\sqrt{[6(a^2+b^2+c^2)+24u\sqrt{3}]}}.$$

Similarly, we find

$$\text{OE} = \frac{(a^2+c^2-b^2)\sqrt{3}+4u}{\sqrt{[6(a^2+b^2+c^2)+24u\sqrt{3}]}}, \quad \text{OD} = \frac{(b^2+c^2-a^2)\sqrt{3}+4u}{\sqrt{[6(a^2+b^2+c^2)+24u\sqrt{3}]}}.$$

From the construction we have OC $= \frac{1}{2}$(OF + OE − OD), OB $= \frac{1}{2}$(OF + OD − OE); hence, since the angle BOC $= 120^\circ$, we have

$$\text{BC} = \sqrt{[(\text{OC})^2+(\text{OB})^2+\text{OC.OB}]}, = \tfrac{1}{2}\sqrt{[3(\text{OF})^2+(\text{OE}-\text{OD})^2]},$$

$$= \tfrac{1}{4}\sqrt{\{6a^2+6b^2-2c^2-8\sqrt{[3s(s-a)(s-b)(s-c)]}\}},$$

and by symmetry,

$$\text{AC} = \tfrac{1}{4}\sqrt{\{6a^2+6c^2-2b^2-8\sqrt{[3s(s-a)(s-b)(s-c)]}\}},$$

$$\text{AB} = \tfrac{1}{4}\sqrt{\{6b^2+6c^2-2a^2-8\sqrt{[3s(s-a)(s-b)(s-c)]}\}}.$$

The solution in Vol. XXXII of the *Educational Times Reprint*, p. 60, (of Quest. 5811,) is mine, although credited (by some mistake I suppose) to others.

II.—Solution by William Hoover, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.

Let ABC be the required triangle, and DEF the triangle formed by joining the remote vertices of the equilateral triangles constructed on the sides AB, AC, BC. Draw the circumscribing circles of the equilateral triangles; they will intersect in the point O. Draw DC, BE and AF; they will intersect in O and be equal, and the angles DOE, EOF and FOD will each be equal to 120°.

Put EF $= a$, FD $= b$ and DE $= c$. Let FO $= u$, EO $= v$ and DO $= w$. Then, since $\cos 120^\circ = -\frac{1}{2}$, we have

$$u^2+uv+v^2 = a^2 \,\ldots\ldots (1), \qquad u^2+uw+w^2 = b^2 \,\ldots\ldots (2),$$

$$v^2+vw+w^2 = c^2 \,\ldots\ldots (3).$$

These equations are the same as (1), (2) and (3) in the solution of Problem 123, No. 4, p. 105; therefore

$$u = \frac{a^2+b^2-c^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}},$$

$$v = \frac{a^2+c^2-b^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}},$$

$$w = \frac{b^2+c^2-a^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

Now let CO $= x$, BO $= y$ and AO $= z$. Then, since each of the angles at O $= 60^\circ$, we have

$$x^2-xu+u^2 = y^2-yu+u^2, \text{ or } x^2-y^2 = u(x-y); \text{ whence } x+y=u \,\ldots\ldots (4).$$

Similarly, we find $x+z=v$ .... (5), and $y+z=w$ .... (6).

$$\therefore x = \tfrac{1}{2}(u+v-w), = \frac{3a^2-b^2-c^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}},$$

$$y = \tfrac{1}{2}(u+w-v), = \frac{3b^2-a^2-c^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}},$$

$$z = \tfrac{1}{2}(v+w-u), = \frac{3c^2-a^2-b^2 \pm \frac{1}{3}\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}}{\sqrt{\{2(a^2+b^2+c^2)\pm 2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}}.$$

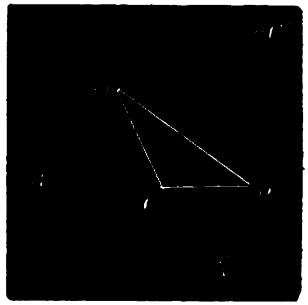
From the triangles AOB, AOC and BOC we have

$$\text{AB} = \sqrt{(y^2+yz+z^2)}, = \tfrac{1}{4}\sqrt{\{6b^2+6c^2-2a^2-2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}},$$

$$\text{AC} = \sqrt{(x^2+xz+z^2)}, = \tfrac{1}{4}\sqrt{\{6a^2+6c^2-2b^2-2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}},$$

$$\text{BC} = \sqrt{(x^2+xy+y^2)}, = \tfrac{1}{4}\sqrt{\{6a^2+6b^2-2c^2-2\sqrt{[12a^2b^2-3(a^2+b^2-c^2)^2]}\}}.$$

III.—Solution by J. F. W. SCHEFFER, Professor of Mathematics and German, Mercersburg College, Mercersburg, Pa.



Denote the sides of the required triangle ABC by x, y, z; and those of the given triangle A′B′C′ by a, b, c; and the angles of the former by A, B, C.

In the triangle AB′C′, we have $a^2 = y^2 + z^2 + 2yz\cos(60^\circ - A)$.

Since $\cos(60^\circ - A) = \cos 60^\circ \cos A + \sin 60^\circ \sin A$, we have

$$2yz\cos(60^\circ - A) = yz\cos A + \sqrt{3}\,yz\sin A = yz\cos A + 2F\sqrt{3},$$

denoting the area of the triangle ABC by F. But since

$$\cos A = \frac{y^2+z^2-x^2}{2yz}, \text{ we have } 2a^2 = 3(y^2+z^2) - x^2 + 4F\sqrt{3} \ldots . (1),$$

and by analogy,

$$2b^2 = 3(x^2+z^2) - y^2 + 4F\sqrt{3} \ldots . (2),\ 2c^2 = 3(x^2+y^2) - z^2 + 4F\sqrt{3} \ldots\ (3).$$

From these we derive by subtraction,

$$\tfrac{1}{4}(a^2-b^2) = y^2 - x^2, \text{ whence } y^2 = \tfrac{1}{4}(a^2-b^2+2x^2); \text{ and } \tfrac{1}{4}(a^2-c^2) = z^2 - x^2, \text{ whence } z^2 = \tfrac{1}{4}(a^2-c^2+2x^2).$$

Substituting these values of y^2 and z^2 in the well-known formula

$$16F^2 = 2(x^2y^2 + x^2z^2 + y^2z^2) - x^4 - y^4 - z^4$$

we get after some reductions which present no difficulties,

$$8F = \sqrt{[12x^4 + 4(2a^2 - b^2 - c^2)x^2 - (b^2-c^2)^2]}.$$

Substituting this value and those of y^2 and z^2 in (1), we obtain a quadratic equation whose root is

$$x^2 = \tfrac{1}{8}[3(b^2+c^2) - a^2 \pm \sqrt{(2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4)}\sqrt{3}].$$

Denoting the area of the given triangle A′B′C′ by $\triangle$, we have, choosing the lower sign,

$$x = \tfrac{1}{4}\sqrt{[6(b^2+c^2) - 2a^2 - 8\triangle\sqrt{3}]}, \text{ and by analogy } y = \tfrac{1}{4}\sqrt{[6(a^2+c^2) - 2b^2 - 8\triangle\sqrt{3}]},$$

$$z = \tfrac{1}{4}\sqrt{[6(a^2+b^2) - 2c^2 - 8\triangle\sqrt{3}]}.$$

Solved also by the *Proposer*, *George Eastwood* and Professor *Edmunds*.

179.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

In a series of rational right-angled triangles where every hypothenuse is unity, each leg of the nth triangle contains n decimal places. Find the legs of the first 25 triangles, and those of the 50th one.

Solution by the PROPOSER.

Three conditions must be fulfilled in the solution of this example: (1) the square of the hypothenuse must equal the sum of the squares of the legs of every triangle; this condition is satisfied by writing $2rs$, $r^2 - s^2$ and $r^2 + s^2$ as the general expressions for the sides of any triangle in the series: (2) the problem requires that every *hypothenuse* shall be unity; this condition is met by dividing the above expressions by $r^2 + s^2$, and our general expressions become $\frac{2rs}{r^2+s^2}$, $\frac{r^2-s^2}{r^2+s^2}$ and 1: the last condition (3) insists upon each leg of the nth triangle containing n decimal places; this demand is complied with by making $r^2 + s^2 = 5^n$.

Since $r^2 + s^2 = 5^n$, it is plain that $(2r+s)^2 + (2s-r)^2 = 5^{n+1}$ and $(2rs)^2 + (r^2-s^2)^2 = 5^{2n}$. We now have a key by which the different values of r and s for the several triangles may be obtained.

Setting out with the values $r = 2$, $s = 1$ we find the successive values to be:

(2) $r = 3$, $s = 4$; **(3)** $r = 11$, $s = 2$; **(4)** $r = 24$, $s = 7$; **(5)** $r = 41$, $s = 38$; **(6)** $r = 44$, $s = 117$;
(7) $r = 29$, $s = 278$; **(8)** $r = 336$, $s = 527$; **(9)** $r = 718$, $s = 1199$; **(10)** $r = 237$, $s = 3116$;
(11) $r = 6469$, $s = 2642$; **(12)** $r = 11753$, $s = 10296$; **(13)** $r = 8839$, $s = 33802$;
(14) $r = 16124$, $s = 76443$; **(15)** $r = 136762$, $s = 103691$; **(16)** $r = 164833$, $s = 354144$;
(17) $r = 873121$, $s = 24478$, **(18)**; $r = 922077$, $s = 1721764$; **(19)** $r = 2521451$, $s = 3565918$;
(20) $r = 1476984$, $s = 9653287$; **(21)** $r = 6699319$, $s = 20783558$; **(22)** $r = 34182196$, $s = 34867797$;
(23) $r = 35553398$, $s = 103232189$; **(24)** $r = 32125393$, $s = 242017776$;
(25) $r = 306268562$, $s = 451910159$; &c.

The legs of the first 25 triangles therefore are:

(1) .6, .8; **(2)** .28, .96; **(3)** .352, .936; **(4)** .5376, .8432; **(5)** .07584, .99712; **(6)** .658944, .752192;
(7) .2063872, .9784704; **(8)** .12197248, .99660864; **(9)** .472103424, .881543168;
(10) .1512431616, .9884965888; **(11)** .70005137408, .71409248256; **(12)** .131585609728, .991304810496;
(13) .8719952142336, .4895143985152; **(14)** .40388753227776, .91480864735232;
(15) .225775162589184, .974179437248512; **(16)** .7651277924204544, .6438784522452992;

(17) .99842930528354304, .05602608634396672; **(18)** .554236714094952448, .832359096033214464;

(19) .3333452483696001024, .9428043288958906368; **(20)** .29900669864185430016, .95425101213847257088;

(21) .583996790525665476608, .811755966196566982656;

(22) .0198561472974078083072, .9998028472726528720896;

(23) .61576662620151796989952, .78792858943967761268736;

(24) .260882895830831308210176, .965370454625020943532032;

(25) .3705159561103475195510784, .9288261011985155397517312.

Legs of 50th triangle,

.66896978189162877470704761248853587979889045886816, .72543585265587006118086083308004714058466679410688.

Legs of 100th triangle,

.0525143522857148180104847694808452054555156688112241412889998780608257799869594195547888576197987966668,

.9066201694337376110.2128825082820175422268914688886146690990002083598817489178816577961236128571255177216.

180.—Proposed by Dr. JOHN BELL, Manchester, Hillsborough County, New Hampshire.

One-third of all the apples on a certain tree are rotten, and one-fourth of all the apples on the same tree are wormy. What are the respective chances that an apple taken at random from the tree will be (1) sound, (2) rotten, (3) wormy, (4) both rotten and wormy?

I.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

1. The least part of apples sound $= 1-(\frac{1}{3}+\frac{1}{4}) = \frac{5}{12}$; greatest part $= 1-\frac{1}{3} = \frac{2}{3}$. All parts between these limits are equally likely, hence the chance of taking a sound apple is $\frac{1}{2}(\frac{2}{3}+\frac{5}{12}) = \frac{13}{24}$.

2. The least part of apples rotten only $= \frac{1}{3}-\frac{1}{4} = \frac{1}{12}$; greatest part $= \frac{1}{3}$; therefore the chance of taking an apple that is rotten only $= \frac{1}{2}(\frac{1}{12}+\frac{1}{3}) = \frac{5}{24}$.

3. The least part of apples wormy only $= 0$; greatest part $= \frac{1}{4}$; therefore chance of taking an apple that is wormy only $= \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$.

4. The least part of apples both rotten and wormy $= 0$; greatest part $= \frac{1}{4}$; therefore the chance of taking an apple that is both rotten and wormy $= \frac{1}{4}\times\frac{1}{2} = \frac{1}{8}$.

II.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society Erie, Erie County, Pa.

Let $12n$ = number of apples on the tree. Then $4n$ = number rotten, and $3n$ = number wormy.

Let p = probability of taking a sound apple at random from the tree, p_1 = probability that the apple taken is rotten only, p_2 = the probability that it is wormy only, and p_3 = the probability that it is both rotten and wormy.

1. The number of sound apples can not be less than $12n-(4n+3n) = 5n$, nor more than $12n-4n = 8n$, and all numbers from $5n$ to $8n$ are equally likely.

If the tree contains $5n$ sound apples the probability of getting one of them is $\frac{5n}{12n}$; but the probability that the tree contains $5n$ sound apples, or any other number from $5n$ to $8n$, is $\frac{1}{(8n-5n)+1} = \frac{1}{3n+1}$, therefore the probability that the tree contains $5n$ sound apples and one will be taken is $\frac{5n}{12n}\times\frac{1}{3n+1}$.

The probability that the tree contains $5n+1$ sound apples and one will be taken is $\frac{5n+1}{12n}\times\frac{1}{3n+1}$.

Similarly, the respective probabilities that the tree contains $5n+2$, $5n+3$, ... $8n$ sound apples, and one of them will be taken, are $\frac{5n+2}{12n(3n+1)}$, $\frac{5n+3}{12n(3n+1)}$, $\frac{5n+4}{12n(3n+1)}$, ... $\frac{8n}{12n(3n+1)}$.

The total probability of obtaining a sound apple at random from the tree is the sum of these separate probabilties.

$$\therefore p = \frac{5n+(5n+1)+(5n+2)+(5n+3)+\ldots\ldots+8n}{12n(3n+1)} = \frac{(5n+8n)\times\frac{1}{2}(3n+1)}{12n\times(3n+1)} = \frac{13}{24}.$$

2. The least number of apples rotten only $= 4n-3n = n$, and the greatest number $= 4n$.

$$\therefore p_1 = \frac{n+(n+1)+(n+2)+(n+3)+\ldots\ldots+4n}{12n(3n+1)} = \frac{(n+4n)\times\frac{1}{2}(3n+1)}{12n(3n+1)} = \frac{5}{24}.$$

3. The least number of apples wormy only $= 0$, and the greatest number $= 3n$.

$$\therefore p_2 = \frac{0+1+2+3+\ldots\ldots+3n}{12n(3n+1)} = \frac{3n\times\frac{1}{2}(3n+1)}{12n(3n+1)} = \tfrac{1}{8}.$$

4. The least number of apples both rotten and wormy $= 0$, and the greatest number $= 3n$.

$$\therefore p_3 = \frac{0+1+2+3+\ldots\ldots+3n}{12n(3n+1)} = \frac{3n\times\frac{1}{2}(3n+1)}{12n(3n+1)} = \tfrac{1}{8}.$$

III.—Solution by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

Put $\frac{a}{b}$ = part rotten and $\frac{c}{d}$ = part wormy. Let p = probability that an apple taken at random from the tree is sound, p_1 = the probability that it is rotten only, p_2 = the probability that it is wormy only and p_3 = the probability that it is both rotten and wormy.

Then we have $p_1+p_3 = \frac{a}{b}$........(1), $p_2+p_3 = \frac{c}{d}$........(2), $p+p_1+p_2+p_3 = 1$........(3).

From (1), (2) and (3), $p_2 = 1-\frac{a}{b}-p$.................(4), $p_1 = 1-\frac{c}{d}-p$.................(5).

1. When $\frac{a}{b}+\frac{c}{d}<1$, the least part of apples sound is $1-\left(\frac{a}{b}+\frac{c}{d}\right)$, and the greatest part is $1-\frac{a}{b}$ if $\frac{a}{b}>\frac{c}{d}$; all parts between these limits are equally likely,

$$\therefore p = \tfrac{1}{2}\left[1-\left(\frac{a}{b}+\frac{c}{d}\right)+\left(1-\frac{a}{b}\right)\right] = 1-\tfrac{1}{2}\left(\frac{2a}{b}+\frac{c}{d}\right) = \tfrac{13}{24}.$$

From (3), (4) and (5) we now readily find, by common rules,

$$p_1 = \frac{a}{b}-\frac{c}{2d} = \tfrac{5}{24}, \qquad p_2 = \frac{c}{2d} = \tfrac{1}{8}, \qquad p_3 = \frac{c}{2d} = \tfrac{1}{8}.$$

2. If $\frac{a}{b}<\frac{c}{d}$, the greatest part of apples sound is $1-\frac{c}{d}$, and

$$p = 1-\tfrac{1}{2}\left(\frac{a}{b}+\frac{2c}{d}\right), \qquad p_1 = \frac{a}{2b}, \qquad p_2 = \frac{c}{d}-\frac{a}{2b}, \qquad p_3 = \frac{a}{2b}.$$

3. When $\frac{a}{b}+\frac{c}{d} = 1$ or >1, the least part of apples sound is 0, and the greatest part is $1-\frac{a}{b}$ if $\frac{a}{b}>\frac{c}{d}$ and $1-\frac{c}{d}$ if $\frac{a}{b}<\frac{c}{d}$; therefore in this case

$$p = \tfrac{1}{2}\left(1-\frac{a}{b}\right) \text{ when } \frac{a}{b}>\frac{c}{d} \text{ and } = \tfrac{1}{2}\left(1-\frac{c}{d}\right) \text{ when } \frac{a}{b}<\frac{c}{d}, \quad p_1 = \tfrac{1}{2}\left(1+\frac{a}{b}-\frac{2c}{d}\right) \text{ or } \tfrac{1}{2}\left(1-\frac{c}{d}\right),$$

$$p_2 = \tfrac{1}{2}\left(1-\frac{a}{b}\right) \text{ or } \tfrac{1}{2}\left(1+\frac{c}{d}-\frac{2a}{b}\right), \qquad p_3 = \tfrac{1}{2}\left(\frac{a}{b}+\frac{2c}{d}-1\right) \text{ or } \tfrac{1}{2}\left(\frac{2a}{b}+\frac{c}{d}-1\right).$$

Solved also in an elegant manner by *Charles H. Kummell* of the U. S. Coast and Geodetic Survey, Washington, D. C.

181.—Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

A polygon is both inscriptible and circumscriptible, radius of inscribed circle r, of circumscribed circle R and distance between the centers of these circles d; then if the polygon be a triangle,

$$\frac{1}{R+d}+\frac{1}{R-d} = \frac{1}{r};$$

if the polygon be a quadrilateral, $$\frac{1}{(R+d)^2}+\frac{1}{(R-d)^2} = \frac{1}{r^2}.$$

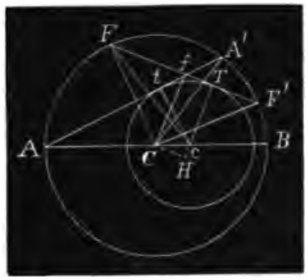
What is the corresponding relation for a pentagon?

Solution by CHARLES H. KUMMELL, U. S. Coast and Geodetic Survey Office, Washington, D. C.

The general solution of this problem can, I believe, only be effected by means of Inverse Elliptic Functions, as has been done by the illustrious Jacobi.—(See Durege, Elliptische Functionen, pp. 164—186.)

I shall condense out of Jacobi's highly interesting discussion that which pertains directly to the problem with some modifications of my own.

Lemma.—If there are two circles, one wholly within the other and the diameter AB $= 2R$ is drawn through their centers C, c, whose distance C$c = d$, and if the radius of the inner circle $= r$; then if from A and likewise from any point F on the outer circle chords AA′ and FF′ are drawn touching the inner circle, and arc AA′ $= 2\alpha$, arc AF $= 2\varphi$, arc AF′ $= 2\varphi'$, we have



$$\int_\phi^{\phi'} \frac{d\varphi}{\Delta\varphi} = \int_0^\alpha \frac{d\varphi}{\Delta\varphi} \quad\quad (1)$$

where $\Delta\varphi = \sqrt{(1-k^2\sin^2\varphi)}$ and the modulus $k = \dfrac{2\sqrt{(Rd)}}{\sqrt{[(R+d)^2-r^2]}}$ ..(2).

If T is the point of tangency of chord FF′ then

$$\text{FT} = \sqrt{[(\text{F}c)^2-r^2]} = \sqrt{(R^2+d^2+2Rd\cos2\varphi-r^2)}$$
$$= \sqrt{[(R+d)^2-r^2-4Rd\sin^2\varphi]} = a\Delta\varphi \quad\quad (3)$$

if we denote the tangent $\quad \text{A}t = \sqrt{[(R+d)^2-r^2]}$ by a ....(4).

Similarly we have $\quad \text{F}'\text{T} = a\Delta\varphi'$ ....(3′) $\quad$ and $\quad \text{A}'t = a\Delta\alpha$ ....(3″).

Draw CH perpendicular to cT; then

$$\text{C}f = \text{HT} = R\cos(\varphi'-\varphi) \quad (5), \quad\quad c\text{H} = d\cos(\varphi'+\varphi) \quad (6);$$

therefore
$$c\text{T} = r = R\cos(\varphi'-\varphi) + d\cos(\varphi'+\varphi) \quad\quad (7).$$

We have
$$\text{F}f = \tfrac{1}{2}(\text{FT}+\text{F}'\text{T}) = \tfrac{1}{2}a(\Delta\varphi'+\Delta\varphi) = R\sin(\varphi'-\varphi) \quad\quad (8),$$
$$\text{CH} = \tfrac{1}{2}(\text{FT}-\text{F}'\text{T}) = \tfrac{1}{2}a(\Delta\varphi-\Delta\varphi') = d\sin(\varphi'+\varphi) \quad\quad (9);$$

$$\therefore \frac{\Delta\varphi+\Delta\varphi'}{\sin(\varphi'-\varphi)} = \frac{2R}{a} \quad (10), \quad\quad \frac{\Delta\varphi-\Delta\varphi'}{\sin(\varphi'+\varphi)} = \frac{2d}{a} \quad (11).$$

On the chord AA′ we have similarly, placing $\varphi = 0$, $\varphi' = \alpha$,

$$\frac{1+\Delta\alpha}{\sin\alpha} = \frac{2R}{a} \quad (10'), \quad\quad \frac{1-\Delta\alpha}{\sin\alpha} = \frac{2d}{a} \quad (11'); \quad\quad \therefore d = \left(\frac{1-\Delta\alpha}{1+\Delta\alpha}\right)R \quad (12).$$

Also,
$$r = (R+d)\cos\alpha = \frac{2R\cos\alpha}{1+\Delta\alpha} \quad\quad (13).$$

These values being used in (7) we obtain, clearing of fractions,

$$\cos\alpha = \tfrac{1}{2}(1+\Delta\alpha)\cos(\varphi'-\varphi) + \tfrac{1}{2}(1-\Delta\alpha)\cos(\varphi'+\varphi) = \cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\Delta\alpha \quad (14).$$

In a spherical triangle in which φ' is an exterior angle and the sides x, x' and A are respectively opposite the angles φ, $\pi-\varphi'$ and α we have $\quad \cos\alpha = \cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'\cos A$ ....(14′).

But if we assume
$$\frac{\sin x}{\sin\varphi} = \frac{\sin x'}{\sin\varphi'} = \frac{\sin A}{\sin\alpha} = k = \frac{2\sqrt{(Rd)}}{a} \quad\quad (15)$$

we have $\quad \cos x = \Delta\varphi, \quad\quad \cos x' = \Delta\varphi', \quad\quad \cos A = \Delta\alpha$ ....(16)

and (14′) becomes identical with (14), and as I have proved in that case (*Analyst*, Vol. V, pp. 17, 18) we have

$$\int_\phi^{\phi'} \frac{d\varphi}{\Delta\varphi} = \int_0^\alpha \frac{d\varphi}{\Delta\varphi} \quad\quad (17).$$

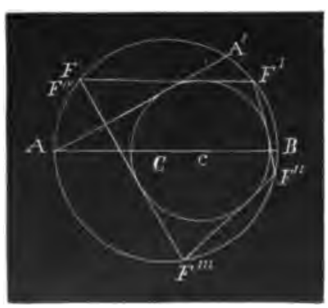
The integral of this equation is either (14) or the forms

$$\sin\alpha = \frac{\sin\varphi'\cos\varphi\Delta\varphi - \sin\varphi\cos\varphi'\Delta\varphi'}{1-k^2\sin^2\varphi\sin^2\varphi'}, \quad\quad \cos\alpha = \frac{\cos\varphi'\cos\varphi + \sin\varphi'\Delta\varphi'\sin\varphi\Delta\varphi}{1-k^2\sin^2\varphi\sin^2\varphi'},$$

$$\Delta\alpha = \frac{\Delta\varphi'\Delta\varphi + k^2\sin\varphi'\cos\varphi'\sin\varphi\cos\varphi}{1-k^2\sin^2\varphi\sin^2\varphi'}$$

which are given l. c. or in any treatise on Elliptic Functions.

If arc AF $= 2\varphi$, arc AF′ $= 2\varphi'$, arc AF″ $= 2\varphi''$, etc., and arc AA′ $= 2\alpha$, then we have by (17)



$$\int_0^\alpha \frac{d\varphi}{\Delta\varphi} = \int_\phi^{\phi'} \frac{d\varphi}{\Delta\varphi} \quad\quad (17)$$
$$= \int_{\phi'}^{\phi''} \frac{d\varphi}{\Delta\varphi} \quad\quad (17')$$
$$\cdots\cdots\cdots\cdots$$
$$= \int_{\phi^{(n-1)}}^{\phi^{(n)}} \frac{d\varphi}{\Delta\varphi} \quad\quad (17^{(n-1)});$$
$$\therefore n\int_0^\alpha \frac{d\varphi}{\Delta\varphi} = \int_\phi^{\phi^{(n)}} \frac{d\varphi}{\Delta\varphi} \quad\quad (18).$$

If $\varphi^{(n)} = \pi + \varphi$ the polygon will close and we have

$$n\int_0^{\alpha} \frac{d\varphi}{\Delta\varphi} = \int_{\phi}^{\pi+\phi} \frac{d\varphi}{\Delta\varphi} = \int_0^{\pi} \frac{d\varphi}{\Delta\varphi} = 2\int_0^{\frac{1}{2}\pi} \frac{d\varphi}{\Delta\varphi} = 2K \qquad \text{(19)},$$

(using Jacobi's notation for the complete elliptic integral of the first species).

We have then

$$\text{arc } \tfrac{1}{2}\text{AA}' = \alpha = \text{am}\, \frac{2K}{n} \left(\text{amplitudo } \frac{2K}{n}\right) \qquad \text{(20)}.$$

This equation or (19) evidently gives implicite the radius of the inner circle if its center is given such that a polygon of n sides may be inscriptible in the outer and circumscriptible to the inner circle.

The integrals of equations (17) are by (14) and (20) for $\varphi = 0$

$$\cos\alpha = \cos \text{am}\, \frac{2}{n} K = \cos \text{am}\, \frac{2}{n} K \qquad \text{(21)},$$

$$= \cos \text{am}\, \frac{4}{n} K \cos \text{am}\, \frac{2}{n} K + \sin \text{am}\, \frac{4}{n} K \sin \text{am}\, \frac{2}{n} K \Delta \text{am}\, \frac{2}{n} K \qquad \text{(21}'\text{)},$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \cos \text{am}\, 2K \cos \text{am}\, 2\left(1 - \frac{1}{n}\right) K + \sin \text{am}\, 2K \sin \text{am}\, 2\left(1 - \frac{1}{n}\right) K \Delta \text{am}\, \frac{2}{n} K \qquad \text{(21}^{(n-1)}\text{)};$$

or, writing $\quad C_{\frac{s}{n}} = \cos \text{am}\, \frac{s}{n} 2K, \qquad S_{\frac{s}{n}} = \sin \text{am}\, \frac{s}{n} 2K, \qquad D_{\frac{1}{n}} = \Delta \text{am}\, \frac{2}{n} K,$

$$C_{\frac{1}{n}} = C_{\frac{2}{n}} C_{\frac{1}{n}} + S_{\frac{2}{n}} S_{\frac{1}{n}} D_{\frac{1}{n}} = C_{\frac{3}{n}} C_{\frac{2}{n}} + S_{\frac{3}{n}} S_{\frac{2}{n}} D_{\frac{1}{n}} = \ldots\ldots$$

$$= C_1 C_{\frac{n-1}{n}} + S_1 S_{\frac{n-1}{n}} D_{\frac{1}{n}} \qquad \text{(21}', 21'', \ldots 21^{(n-1)}\text{)}.$$

By the relations $\sin \text{am}\,(2K - \epsilon) = \sin \text{am}\, \epsilon \quad \therefore \quad S_{\frac{s}{n}} = S_{1-\frac{s}{n}} \qquad \text{(22)},$

$$\cos \text{am}\,(2K - \epsilon) = -\cos \text{am}\, \epsilon \quad \therefore \quad C_{\frac{s}{n}} = -C_{1-\frac{s}{n}} \qquad \text{(23)},$$

equations (21′) become if n is even

$$C_{\frac{1}{n}} = C_{\frac{2}{n}} C_{\frac{1}{n}} + S_{\frac{2}{n}} S_{\frac{1}{n}} D_{\frac{1}{n}} = C_{\frac{3}{n}} C_{\frac{2}{n}} + S_{\frac{3}{n}} S_{\frac{2}{n}} D_{\frac{1}{n}} = \ldots\ldots = \ldots\ldots \qquad \text{(21}', 21''\text{)}$$

$$= C_{\frac{1}{2}} C_{\frac{1}{2}-\frac{1}{n}} + S_{\frac{1}{2}} S_{\frac{1}{2}-\frac{1}{n}} D_{\frac{1}{n}} = S_{\frac{1}{2}-\frac{1}{n}} D_{\frac{1}{n}} \qquad \text{(21}^{\frac{1}{2}(n-2)}\text{)}$$

because $C_{\frac{1}{2}} = \cos \text{am}\, K = 0$ and $S_{\frac{1}{2}} = 1$,

$$= C_{\frac{1}{2}+\frac{1}{n}} C_{\frac{1}{2}} + S_{\frac{1}{2}+\frac{1}{n}} S_{\frac{1}{2}} D_{\frac{1}{n}} = S_{\frac{1}{2}-\frac{1}{n}} D_{\frac{1}{n}} \text{ by (22)}, \qquad \text{(21}^{\frac{1}{2}(n+2)}\text{)}$$

$$= C_{\frac{1}{2}+\frac{2}{n}} C_{\frac{1}{2}+\frac{1}{n}} + S_{\frac{1}{2}+\frac{2}{n}} S_{\frac{1}{2}+\frac{1}{n}} D_{\frac{1}{n}} = C_{\frac{1}{2}-\frac{2}{n}} C_{\frac{1}{2}-\frac{1}{n}} + S_{\frac{1}{2}-\frac{2}{n}} S_{\frac{1}{2}-\frac{1}{n}} D_{\frac{1}{n}} \qquad \text{(21}^{\frac{1}{2}(n+4)}\text{)}$$

by (22) and (23),

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= C_{1-\frac{1}{n}} C_{1-\frac{2}{n}} + S_{1-\frac{1}{n}} S_{1-\frac{2}{n}} D_{\frac{1}{n}} = C_{\frac{1}{n}} C_{\frac{2}{n}} + S_{\frac{1}{n}} S_{\frac{2}{n}} D_{\frac{1}{n}} \qquad \text{(21}^{(n-2)}\text{)}$$

by (22) and (23),

$$= C_1 C_{1-\frac{1}{n}} + S_1 S_{1-\frac{1}{n}} D_{\frac{1}{n}} = C_{\frac{1}{n}} \text{ because } C_1 = 1,\ S_1 = 0 \qquad \text{(21}^{(n-1)}\text{)}.$$

There are then only $\frac{1}{2}n$ really different equations. If n is odd there will be one middle equation, viz:

$$C_{\frac{1}{n}} = C_{\frac{n+1}{2n}} C_{\frac{n-1}{2n}} + S_{\frac{n+1}{2n}} S_{\frac{n-1}{2n}} D_{\frac{1}{n}} = -C^2_{\frac{n-1}{2n}} + S^2_{\frac{n-1}{2n}} D_{\frac{1}{n}} \qquad \text{(21}^{\frac{1}{2}(n-1)}\text{)},$$

by (22) and (23).

Because $C^2_{\frac{2}{n}} + S^2_{\frac{2}{n}} = 1$ we can express $C_{\frac{2}{n}}$ and $S_{\frac{2}{n}}$ from (21′), then $C_{\frac{3}{n}}$ and $S_{\frac{3}{n}}$ from (21″), etc., in terms of $C_{\frac{1}{n}}$, $S_{\frac{1}{n}}$ and $D_{\frac{1}{n}}$. If n is even $(21^{\frac12(n-2)})$, and if n is odd $(21^{\frac12(n-1)})$ becomes an equation between $C_{\frac{1}{n}}$, $S_{\frac{1}{n}}$ and $D_{\frac{1}{n}}$; that is, between $\cos\alpha = \cos\operatorname{am}\frac{2}{n}K$, $\sin\alpha$ and $\Delta\alpha$. But solving (12) for $\Delta\alpha$ we obtain $\Delta\alpha = D_{\frac{1}{n}} = \frac{R-d}{R+d}$, $\quad\therefore\quad \cos\alpha = C_{\frac{1}{n}} = \frac{r}{R+d}$ by (13) .... (24), and $\sin\alpha = S_{\frac{1}{n}} = \frac{a}{R+d}$ by (4). We have, finally, an equation in R, d, r and $a = \sqrt{[(R+d)^2 - r^2]}$.

1. For $n = 3$ we have

$$C_{\frac13} = C_{\frac23}C_{\frac13} + S_{\frac23}S_{\frac13}D_{\frac13} = -C^2_{\frac13} + S^2_{\frac13}D_{\frac13}$$

or

$$\frac{r}{R+d} = -\frac{r^2}{(R+d)^2} + \left(\frac{a^2}{(R+d)^2}\right)\left(\frac{R-d}{R+d}\right) = \frac{R-d}{R+d} - \left(\frac{2R}{R+d}\right)\left(\frac{r^2}{(R+d)^2}\right);$$

that is, $r = R - d - \frac{2Rr^2}{(R+d)^2}$. Solving this quadratic we find $r = \frac{R+d}{4R}[-R-d \pm (3R-d)]$. We have to reject the lower sign because $r = -(R+d)$ makes the radius of the inner circle greater than that of the outer and negative which is against the supposition. There remains only

$$r = \frac{R^2-d^2}{2R}, \qquad \text{or} \qquad \frac{1}{r} = \frac{1}{R+d} + \frac{1}{R-d} \text{ .... (25).}$$

2. For $n = 4$ we have

$$C_{\frac14} = C_{\frac12}C_{\frac14} + S_{\frac12}S_{\frac14}D_{\frac14} = S_{\frac14}D_{\frac14}, \text{ or by (24) } \frac{r}{R+d} = \left(\frac{a}{R+d}\right)\left(\frac{R-d}{R+d}\right).$$

Squaring, $r^2 = \left(\frac{R-d}{R+d}\right)^2[(R+d)^2 - r^2]$; $\therefore$ $\frac{1}{r^2} = \frac{2(R^2+d^2)}{R^2-d^2} = \frac{1}{(R+d)^2} + \frac{1}{(R-d)^2}$ .... (26).

3. For $n = 5$ we have

$$C_{\frac15} = C_{\frac25}C_{\frac15} + S_{\frac25}S_{\frac15}D_{\frac15} = C_{\frac35}C_{\frac25} + S_{\frac35}S_{\frac25}D_{\frac15} = -C^2_{\frac15} + S^2_{\frac15}D_{\frac15}.$$

From the second equation we find

$$C_{\frac25} = \sqrt{\left(\frac{D_{\frac15} - C_{\frac15}}{D_{\frac15} + 1}\right)}, \qquad S_{\frac25} = \sqrt{\left(\frac{C_{\frac15} + 1}{D_{\frac15} + 1}\right)}.$$

Substituting in the first equation we obtain

$$0 = C_{\frac15}\sqrt{(D_{\frac15} + 1)} - C_{\frac15}\sqrt{(D_{\frac15} - C_{\frac15})} - S_{\frac15}D_{\frac15}\sqrt{(C_{\frac15} + 1)},$$

$$= \left(\frac{r}{R+d}\right)\sqrt{\left(\frac{2r}{R+d}\right)} - \left(\frac{r}{R+d}\right)\sqrt{\left(\frac{R-d-r}{R+d}\right)} - \left(\frac{a(R-d)}{(R+d)^2}\right)\sqrt{\left(\frac{R+d+r}{R+d}\right)},$$

$$= r(R+d)[\sqrt{(2r)} - \sqrt{(R-d-r)}] - (R-d)(R+d+r)\sqrt{(R+d-r)} \text{ .... (27).}$$

. Progressing in this manner we find for the hexagon

$$0 = R^2 - d^2 - (R+d)\sqrt{[(R-d)^2 - r^2]} - (R-d)\sqrt{[(R+d)^2 - r^2]} \text{ .... (28);}$$

for the heptagon,

$$0 = r[4Rdr^2 + (R^2-d^2)^2]\sqrt{(2R)} - (R+d)[2(R^2+d^2)r^2 - (R^2-d^2)^2]\sqrt{(R-d-r)}$$
$$- 2r(R+d)(R-d)^2(R+d+r)\sqrt{(R+d-r)} \text{ ...... (29);}$$

for the octagon,

$$0 = r(R-d)[4Rdr^2 + (R^2-d^2)^2] - (R+d)[2(R^2+d^2)r^2 - (R^2-d^2)^2]\sqrt{[(R-d)^2 - r^2]}$$
$$- 2r^2(R-d)(R+d)^2\sqrt{[(R+d)^2 - r^2]} \text{ ...... (30);}$$

and so on.

182.—Proposed by ALEXANDER MACFARLANE, M. A., D. Sc., Edinburg, Scotland.

Prove that

$$\log[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{2}} = \log a + \frac{b}{a}\cos(\alpha-\beta) - \tfrac{1}{2}\frac{b^2}{a^2}\cos 2(\alpha-\beta) + \tfrac{1}{3}\frac{b^3}{a^3}\cos 3(\alpha-\beta) - \text{etc.}$$

I.—Solution by WALTER SIVERLY, Oil City, Venango County, Pennsylvania.

Put $\alpha-\beta=\gamma$ and, applying the well-known formula

$$\log(c+\chi) = \log c + \frac{\chi}{c} - \frac{\chi^2}{2c^2} + \frac{\chi^3}{3c^3} - \frac{\chi^4}{4c^4} + \text{etc.},$$

$$\log(a^2+b^2+2ab\cos\gamma)^{\frac{1}{2}} = \log a + \frac{(b^2+2ab\cos\gamma)}{2a^2} - \frac{(b^2+2ab\cos\gamma)^2}{4a^4} + \frac{(b^2+2ab\cos\gamma)^3}{6a^6} - \text{etc.}$$

Expanding and arranging terms in ascending powers of $\frac{b}{a}$,

$$\log(a^2+b^2+2ab\cos\gamma)^{\frac{1}{2}} = \log a + \frac{b}{a}\cos\gamma - \frac{b^2}{2a^2}(2\cos^2\gamma-1) + \frac{b^3}{3a^3}(4\cos^3\gamma-3\cos\gamma) - \text{etc.},$$

$$= \log a + \frac{b}{a}\cos\gamma - \frac{b^2}{2a^2}\cos 2\gamma + \frac{b^3}{3a^3}\cos 3\gamma - \frac{b^4}{4a^4}\cos 4\gamma + \text{etc.};$$

$$\therefore\ \log[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{2}} = \log a + \frac{b}{a}\cos(\alpha-\beta) - \frac{b^2}{2a^2}\cos 2(\alpha-\beta) + \frac{b^3}{3a^3}\cos 3(\alpha-\beta) - \text{etc.}$$

II.—Solution by W. E. HEAL, Marion, Grant County, Ind.; and WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.

Put $\phi=(\alpha-\beta)\sqrt{-1}$; then, by Euler's Theorem, $2\cos n(\alpha-\beta) = \varepsilon^{n\phi}+\varepsilon^{-n\phi}$.

$$a^2+b^2+2ab\cos(\alpha-\beta) = a^2\left[1+\frac{b^2}{a^2}+\frac{b}{a}\left(\varepsilon^{\phi}+\varepsilon^{-\phi}\right)\right] = a^2\left(1+\frac{b}{a}\varepsilon^{\phi}\right)\left(1+\frac{b}{a}\varepsilon^{-\phi}\right);$$

$$\log[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{2}} = \log a + \tfrac{1}{2}\left[\log\left(1+\frac{b}{a}\varepsilon^{\phi}\right)+\log\left(1+\frac{b}{a}\varepsilon^{-\phi}\right)\right].$$

Developing by the formula $\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \text{etc.}$,

$$\log[a^2+b^2+2ab\cos(\alpha-\beta)]^{\frac{1}{2}} = \log a + \frac{b}{a}\left[\tfrac{1}{2}\left(\varepsilon^{\phi}+\varepsilon^{-\phi}\right)\right] - \tfrac{1}{2}\frac{b^2}{a^2}\left[\tfrac{1}{2}\left(\varepsilon^{2\phi}+\varepsilon^{-2\phi}\right)\right] + \text{etc.},$$

$$= \log a + \frac{b}{a}\cos(\alpha-\beta) - \tfrac{1}{2}\frac{b^2}{a^2}\cos 2(\alpha-\beta) + \tfrac{1}{3}\frac{b^3}{a^3}\cos 3(\alpha-\beta) - \text{etc.}$$

Solved also by Professor *Ludwig* and Professor *Scheffer*.

183.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

A clock which indicates correct time at the level of the ocean is carried to the top of a mountain near by, the ascent occupying h hours. On arriving at the top of the mountain the clock was found to be m minutes too slow. Required the hight of the mountain.

Solution by EDWARD A. BOWSER, Professor of Mathematics and Engineering, Rutgers College, New Brunswick, N. J.

Let a = the hight of the mountain, r = the radius of the earth at the level of the ocean and t the time of ascent to any point P. Put $3600h=b$.

Then from the principles of Mechanics we have $\frac{at}{3600h}\left(=\frac{at}{b}\right)$ = hight of P above the level of the ocean, $\pi\sqrt{\left(\frac{l}{g}\right)}$ = time of vibration, = one second at the level of the ocean by hypothesis.

$$\sqrt{g'} : \sqrt{g} :: r : r+\frac{at}{b};\quad \therefore\quad \frac{1}{\sqrt{g'}} = \frac{1}{\sqrt{g}}\left(1+\frac{at}{br}\right).$$

$\pi\sqrt{\left(\frac{l}{g'}\right)} = \left(1+\frac{at}{br}\right)$ = time of vibration at the point P. $1-\frac{br}{br+at}$ = number of vibrations lost at the point P in one second.

$\therefore \int_0^b\left(1-\frac{br}{br+at}\right)dt$ = number of vibrations lost in ascending the mountain in the time b (= $3600h$ seconds), which = $60m$ by hypothesis.

Hence integrating we have

$$1-\frac{r}{a}\log\left(1+\frac{a}{r}\right)=\frac{60m}{b}.$$

Developing by Maclaurin's Theorem we have

$$\frac{a}{2r^2}-\frac{a^2}{3r^3}+\frac{a^3}{4r^4}-\frac{a^4}{5r^5}+\text{etc.}=\frac{60m}{br}.$$

Reverting this series and restoring the value of b, we have

$$a=2r\left(\frac{m}{60h}\right)+\frac{8r}{3}\left(\frac{m}{60h}\right)^2+\frac{28r}{9}\left(\frac{m}{60h}\right)^3+\frac{464r}{135}\left(\frac{m}{60h}\right)^4+\text{etc.}$$

Solved in a similar manner by Professor *Seitz* and *William Hoover*.

184.—Proposed by Dr. S. H. WRIGHT, M. A., Ph. D., late Mathematical Editor *Yates County Chronicle*, Penn Yan, N. Y.

Given the latitude $=\lambda=42^\circ\ 30'$ N., the sun's declination $=\delta=20^\circ$ N., his radius $=r=16'$, and a vertical wall running S. 10° W., to find when the sun will first shine on the west side of the wall, the points from the sun's north point of ingress and egress on the wall, the altitudes of those points, and of the sun's center.

Solution by E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.

Let Z be the zenith of the given place, N the north celestial pole, ZQ the vertical circle coinciding with the plane of the wall, S the place of the sun's center at the required time, and T the point of contact with ZQ. Draw the arc NM perpendicular to QZ, and produce it till it meets TS produced at P. Join NS and ZS.

Let $ZN=90^\circ-\lambda$, $SN=90^\circ-\delta$, $ST=r$, $SP=90^\circ-r$, $MN=\gamma$, $ZM=\phi$, $PN=\theta$, $ZS=\nu$, $\angle NZM=\beta=10^\circ$, $\angle ZNM=\psi$, $\angle SNP=\mu$, $\angle SPN=MT=\omega$, and $\angle PSN=\rho$.

Then in the triangle MNZ we have $\gamma=\sin^{-1}(\sin\beta\cos\lambda)=7^\circ\ 21'\ 20''$, $\phi=\tan^{-1}(\cos\beta\cot\lambda)=47^\circ\ 3'\ 46''$, and $\psi=\cot^{-1}(\tan\beta\sin\lambda)=83^\circ\ 12'\ 24''$;

$\therefore\ \theta=90^\circ+\gamma=97^\circ\ 21'\ 20''$.

In the triangle NPS we have

$$\mu=2\tan^{-1}\left(\frac{\sin\frac{1}{2}(\theta-r+\delta)\cos\frac{1}{2}(\theta+r+\delta)}{\cos\frac{1}{2}(\theta-r-\delta)\sin\frac{1}{2}(\theta+r-\delta)}\right)^{\frac{1}{2}}=87^\circ\ 1'\ 14'',$$

$$\omega=2\tan^{-1}\left(\frac{\cos\frac{1}{2}(\theta+r+\delta)\sin\frac{1}{2}(\theta+r-\delta)}{\cos\frac{1}{2}(\theta-r-\delta)\sin\frac{1}{2}(\theta-r+\delta)}\right)^{\frac{1}{2}}=69^\circ\ 47'\ 24'',$$

$$\rho=2\tan^{-1}\left(\frac{\sin\frac{1}{2}(\theta-r+\delta)\sin\frac{1}{2}(\theta+r-\delta)}{\cos\frac{1}{2}(\theta-r-\delta)\cos\frac{1}{2}(\theta+r+\delta)}\right)^{\frac{1}{2}}=97^\circ\ 55'\ 43''.$$

Hence the hour angle of the sun $=\mu-\psi=3^\circ\ 48'\ 50''$, and the required time is $\frac{1}{15}(\mu-\psi)=15$ min. $15\frac{1}{3}$ sec. after apparent noon. The distance of the point of ingress from the north point of the sun is $180^\circ-\rho=82^\circ\ 4'\ 17''$, and the altitude of this point is $90^\circ-(\omega-\phi)=67^\circ\ 16'\ 22''$.

In the triangle STZ we have $\nu=\cos^{-1}[(\cos r\cos(\omega-\phi)]=22^\circ\ 43'\ 44''$; hence the altitude of the sun's center is $90^\circ-\nu=67^\circ\ 16'\ 16''$.

For the point of egress we must substitute $-r$ for r in the expressions for ω and ρ; hence we have

$$\omega'=2\tan^{-1}\left(\frac{\cos\frac{1}{2}(\theta-r+\delta)\sin\frac{1}{2}(\theta-r-\delta)}{\cos\frac{1}{2}(\theta+r-\delta)\sin\frac{1}{2}(\theta+r+\delta)}\right)^{\frac{1}{2}}=69^\circ\ 51'\ 48'',$$

$$\rho'=2\tan^{-1}\left(\frac{\sin\frac{1}{2}(\theta+r+\delta)\sin\frac{1}{2}(\theta-r-\delta)}{\cos\frac{1}{2}(\theta+r-\delta)\cos\frac{1}{2}(\theta-r+\delta)}\right)^{\frac{1}{2}}=97^\circ\ 43'\ 58'',$$

which is the distance of the point of egress from the north point of the sun. The altitude of this point is $90^\circ-(\omega'-\phi)=67^\circ\ 11'\ 58''$, the zenith distance of the sun's center is $\nu'=\cos^{-1}[\cos r\cos(\omega'-\phi)]=22^\circ\ 48'\ 8''$, and hence its altitude is $90^\circ-\nu'=67^\circ\ 11'\ 52''$.

Solved also by the *Proposer* and Professor *De Volson Wood*.

185.—Proposed by SAMUEL ROBERTS, M. A., F. R. S., President of the London Mathematical Society, London, England.

Show that if the system of equations

$$ax^2+bxy+cy^2=u^2, \qquad ax^2-bxy-cy^2=v^2,$$

are resoluble by integer values of x, y, u, v, (zero excluded,) not having a common factor greater than unity, successive solutions can be obtained of the same character.

Solution by the PROPOSER.

We suppose a, b, c to be integers and for the satisfied pair of equations we may write

$$A+B+C=u^2, \qquad A-B-C=v^2$$

where A stands for ax_1^2, B for bx_1y_1, and C for cy_1^2.

Then if $\quad Ax^2+Bxy+Cy^2=U^2, \qquad Ax^2-Bxy-Cy^2=V^2$

it may be assumed that

$$x=k^2+l^2, \qquad U=ux+2k(vl-uk), \qquad V=vx-2l(vl-uk)$$

and by substitution

$$Cy^2+B(k^2+l^2)y=[(k^2+l^2)u+2k(vl-uk)]^2-A(k^2+l^2)^2,$$

or $\quad [2Cy+B(k^2+l^2)]^2=4C[(k^2+l^2)u+2k(vl-uk)]^2-(4AC-B^2)(k^2+l^2)^2,$

$$=(B+2C)^2(k^2+l^2)^2+16Cuvkl(k^2+l^2)-32Cuvk^3l-16C(u^2-v^2)k^2l^2,$$

and y is rational and generally integral if

$$[2Cu^2v^2+(B+2C)^2(B+C)]l+(B+2C)^2uvk=0$$

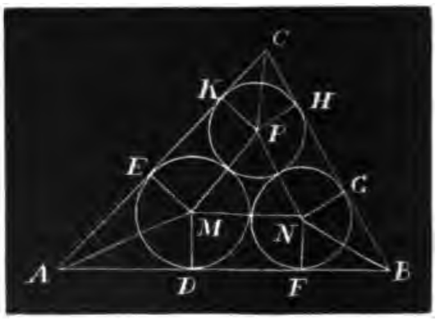
so that

$$y=x+\frac{4uvkl}{B+2C}.$$

186.—Proposed by MARCUS BAKER, U. S. Coast and Geodetic Survey Office, Washington, D. C.

Inscribe in any plane triangle three circles each tangent to two sides and the other two circles.

I.—Solution by E. J. EDMUNDS, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana.



Let M, N, P be the centers of circles inscribed in the angles A, B, C. Draw MD, NF perpendicular to AB.

Assume $AD=x$, $BF=y$, $CH=z$; $a=p\sin^2\alpha$, $b=p\sin^2\beta$, $c=p\sin^2\gamma$, with $p=\frac{1}{2}(a+b+c)$ and a, b, c lengths of sides of given triangle ABC; α, β, γ are then determined.

Let us again assume $\lambda=\frac{1}{2}(\alpha+\beta+\gamma)$; we shall prove that $\quad x=p\sin^2(\lambda-\alpha), \quad y=p\sin^2(\lambda-\beta), \quad z=p\sin^2(\lambda-\gamma).$

In order to do so let us remark that the radii of the two circles tangent to AB are expressed as follows:

$$MD=x\tan\tfrac{1}{2}A, \qquad NF=y\tan\tfrac{1}{2}B.$$

Evidently we have

$$DF=c-x-y=\sqrt{[(MD+NF)^2-(MD-NF)^2]}=2\sqrt{(MD\times NF)}=2\sqrt{(xy\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B)}.$$

But by Trigonometry

$$\tan\tfrac{1}{2}A=\sqrt{\left(\frac{(p-b)(p-c)}{p(p-a)}\right)},\ \tan\tfrac{1}{2}B=\sqrt{\left(\frac{(p-a)(p-c)}{p(p-b)}\right)};\ \therefore\ \tan\tfrac{1}{2}A\tan\tfrac{1}{2}B=1-\frac{c}{p}=\cos^2\gamma,$$

and we have

$$\left.\begin{aligned} x+y+2\cos\gamma\sqrt{(xy)}&=p\sin^2\gamma,\\ x+z+2\cos\beta\sqrt{(xz)}&=p\sin^2\beta,\\ y+z+2\cos\alpha\sqrt{(yz)}&=p\sin^2\alpha.\end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

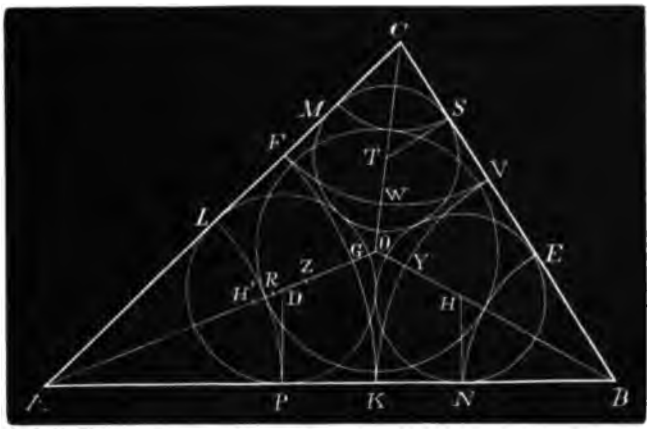
In a circle, radius $=1$, construct a triangle having angles $\lambda-\alpha$, $\lambda-\beta$ and $\pi-\gamma$ (their sum $=\pi$, referring to the value of λ). The sides of that triangle are expressed by $\sin(\lambda-\alpha)$, $\sin(\lambda-\beta)$ and $\sin\gamma$, and we have

$$\left.\begin{aligned} \sin^2(\lambda-\alpha)+\sin^2(\lambda-\beta)+2\cos\gamma\sin(\lambda-\alpha)\sin(\lambda-\beta)&=\sin^2\gamma,\\ \sin^2(\lambda-\alpha)+\sin^2(\lambda-\gamma)+2\cos\beta\sin(\lambda-\alpha)\sin(\lambda-\gamma)&=\sin^2\beta,\\ \sin^2(\lambda-\beta)+\sin^2(\lambda-\gamma)+2\cos\alpha\sin(\lambda-\beta)\sin(\lambda-\gamma)&=\sin^2\alpha;\end{aligned}\right\}\ \ldots\ldots\ldots\text{(II)}$$

and if we take $\quad x=p\sin^2(\lambda-\alpha), \quad y=p\sin^2(\lambda-\beta), \quad z=p\sin^2(\lambda-\gamma)\quad$ (I) and (II) are identical, therefore the problem is solved.

The substance of this solution is taken from Prof. Housel's Introduction a la Geometrie Superieure, page 144.

II.—Solution by Rev. U. Jesse Knisely, Ph. D., Newcomerstown, Tuscarawas County, Ohio.



Designate the sides of ABC, as usual, by a, b and c. Suppose the circles inscribed. Let $DP = x$, $HN = y$, $ST = z$. We know that $BK = \frac{1}{2}(a+c-b)$, $CV = \frac{1}{2}(a+b-c)$, $AK = \frac{1}{2}(b+c-a)$; designate these segments by m, n, p. Also, $OK = r = \sqrt{\left(\frac{(a+b-c)(a-b+c)(b-a+c)}{4(a+b+c)}\right)}$,

$$AO = \sqrt{\left(\frac{bc(b+c-a)}{a+b+c}\right)},\qquad BO = \sqrt{\left(\frac{ac(a+c-b)}{a+b+c}\right)},\qquad CO = \sqrt{\left(\frac{ab(a+b-c)}{a+b+c}\right)}.$$

$$AP = \frac{px}{r},\qquad BN = \frac{my}{r},\qquad CS = \frac{nz}{r},\qquad PK + KN = c - \frac{px+my}{r}.$$

$$(PD-HN)^2 + (PN)^2 = (DH)^2,\quad (HE-ST)^2 + (SE)^2 = (TH)^2,\quad (DL-MT)^2 + (ML)^2 = (TD)^2;$$

or, by substitution,

$$(x-y)^2 + \left(c - \frac{px+my}{r}\right)^2 = (x+y)^2,\qquad (x-z)^2 + \left(b - \frac{px+nz}{r}\right)^2 = (x+z)^2,$$

$$(y-z)^2 + \left(a - \frac{my+nz}{r}\right)^2 = (y+z)^2.$$ Hence we have

$$2\surd(xy) = c - \frac{px+my}{r}\;.....(1),\quad 2\surd(xz) = b - \frac{px+nz}{r}\;.....(2),\quad 2\surd(yz) = a - \frac{my+nz}{r}\;.....(3).$$

From (2) and (3) we have

$$n\surd z + r\surd x = \surd[(rnb - (pn-r^2)x],\qquad n\surd z + r\surd y = \surd[rna - (mn-r^2)y].$$

Hence $$r(\surd x - \surd y) = \surd[rnb - (pn-r^2)x] - \surd[rna - (mn-r^2)y].$$

Squaring, transposing, collecting, and putting for $px+my$ its value $rc - 2r\surd(xy)$, found from (1), we have, since $mn - r^2 = \frac{2a(a+c-b)(a-b+c)}{4(a+b+c)}$ and $pn - r^2 = \frac{2b(b+c-a)(a+b-c)}{4(a+b+c)}$,

$$rn(a+b-c) + (2r^2+2rn)\surd(xy) = 2\sqrt{\left(r^2n^2ab - \frac{2rn^2ab}{a+b+c}(px+my) + \frac{2r^2nabxy}{a+b+c}\right)},$$

$$= 2r\sqrt{\left[ab\left(\frac{2n^3}{a+b+c} + \frac{4n^2\surd(xy)}{a+b+c} + \frac{2nxy}{a+b+c}\right)\right]}$$

after some transformations; hence

$$rn(a+b-c) + (2r^2+2rn)\surd(xy) = 2r\sqrt{\left(\frac{2abn}{a+b+c}[n^2+2n\surd(xy)+xy]\right)}.$$

Therefore $$2n^2 + (2r+2n)\surd(xy) = 2[n+\surd(xy)]\sqrt{\left(\frac{2abn}{a+b+c}\right)},$$

$$2xy = \frac{2n\left[\sqrt{\left(\frac{2abn}{a+b+c}\right)} - n\right]}{r+n-\sqrt{\left(\frac{2abn}{a+b+c}\right)}},\; = \frac{2n\left[\sqrt{\left(\frac{2abn}{a+b+c}\right)} - n\right]\left[r-n+\sqrt{\left(\frac{2abn}{a+b+c}\right)}\right]}{r^2 - \left[\sqrt{\left(\frac{2abn}{a+b+c}\right)} - n\right]^2},$$

$$= \frac{2CV(OC-CV)(r+OC-CV)}{r^2-(OC-CV)^2}\;......................\;......(X).$$

But $\frac{2CV(OC-CV)}{r^2-(OC-CV)^2}=\frac{2CV(OC-CV)}{2CV.OC-(CV)^2-[(OC)^2-r^2]}=\frac{2CV(OC-CV)}{2CV.OC-2(CV)^2}=\frac{OC-CV}{OC-CV}=1$;

hence (X) becomes $2\surd(xy)=PN=r+OC-CV$.

Similarly $2\surd(xz)=ML=r+OB-BK$, $2\surd(yz)=SE=r+OA-AK$.

Now, to find AP, BN and CS:

$AP+BN=c-PN$ ......(4), $AP+CS=b-ML$ ......(5), $BN+CS=a-SE$ ......(6).

Subtract (6) from (5), add to (4) and divide by 2; then

$AP=\frac{1}{2}(b+c-a-ML-PN+SE)$, $=\frac{1}{2}[AK-(OB+r-BK)-(OC+r-CN)+(OA+r-AK)]$,

$=\frac{1}{2}[OA+AK-(OB+BK)-(OC-CV)-r]$,

$=\frac{1}{2}[2OA-(OA-AK)-(OB-BK)-(OC-OV)-r]$,

$=OA-\frac{1}{2}[r+(OA-AK)+(OB-BK)+(OC-CV)]$,

$=OA-\{r-\frac{1}{2}[r-(OA-AK)-(OB-BK)-(OC-CV)]\}$.

So, $BN=OB-\{r-\frac{1}{2}[r-(OA-AK)-(OB-BK)-(OC-CV)]\}$,

$CS=OC-\{r-\frac{1}{2}[r-(OA-AK)-(OB-BK)-(OC-CV)]\}$.

If α, β, γ denote the lines OA, OB, OC and $s=\frac{1}{2}(a+b+c)$,

$AP=\frac{1}{2}(s-r+\alpha-\beta-\gamma)$, $BN=\frac{1}{2}(s-r+\beta-\alpha-\gamma)$, $CS=\frac{1}{2}(s-r+\gamma-\alpha-\beta)$.

From the similar triangles AOK, ADP we have

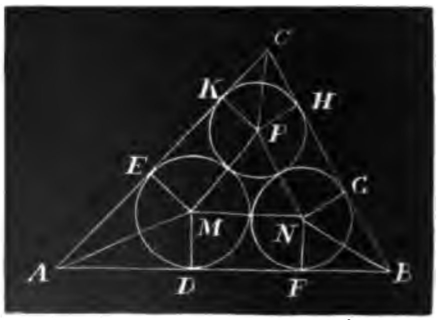
$AK:OK::AP:DP$; or $s-a:r::\frac{1}{2}(s-r+\alpha-\beta-\gamma):x$;

whence $x=\frac{r(s-r+\alpha-\beta-\gamma)}{2(s-a)}$. Similarly, $y=\frac{r(s-r+\beta-\alpha-\gamma)}{2(s-b)}$, $z=\frac{r(s-r+\gamma-\alpha-\beta)}{2(s-c)}$.

Construction.—Let ABC be the triangle, F, V, K the points of contact of the inscribed circle, radius r. From centers A, B, C describe the arcs FK, VK, FV, intersecting AO, BO, CO in the points G, Y, W. From the center O, on OA, take GZ = OW; then ZR = OY. Take OH′ = r, and through the point of bisection of RH′ describe a circle having its center at O. With A, B, C as centers describe arcs touching the circle last drawn and cutting AB, BC in the points P, N and S which will be the points of contact. From these points erect perpendiculars to AB, BC intersecting OA, OB, OC in the points D, H, T; these last points are the centers of the required circles. For

$AP=OA-[r-\frac{1}{2}(r-OW-OY-OG)]$, $=OA-\{r-\frac{1}{2}[r-(OC-CV)-(OA-AK)-(OB-BK)]\}$.

III.—Solution by E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics at the North Missouri State Normal School, Kirksville, Missouri.



Let ABC be the triangle, M, N, P the centers of the circles, D, E, F, G, H, K the points of tangency.

Let $MD=x$, $NF=y$, $PH=z$, and $r=$ the radius of the inscribed circle of the triangle. Then $AD=AE=x\cot\frac{1}{2}A$, $BF=BG=y\cot\frac{1}{2}B$, $CH=CK=z\cot\frac{1}{2}C$, $DF=2\surd(xy)$, $EK=2\surd(xz)$, $GH=2\surd(yz)$, and we have the equations,

$x\cot\frac{1}{2}A+2\surd(xy)+y\cot\frac{1}{2}B=c$ ....(1),

$x\cot\frac{1}{2}A+2\surd(xz)+z\cot\frac{1}{2}C=b$ ....(2),

$y\cot\frac{1}{2}B+2\surd(yz)+z\cot\frac{1}{2}C=a$ ....(3).

By Trigonometry we have

$\frac{\sin\frac{1}{2}B\cos\frac{1}{2}B}{b}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c}$ ....(4), $\frac{\sin\frac{1}{2}A\cos\frac{1}{2}A}{a}=\frac{\sin\frac{1}{2}C\cos\frac{1}{2}C}{c}$ ....(5);

and from these two equations we can deduce the following:

$b(\cot\frac{1}{2}A-\tan\frac{1}{2}B)=c(\cot\frac{1}{2}A-\tan\frac{1}{2}C)$.....(6), $a(\cot\frac{1}{2}B-\tan\frac{1}{2}A)=c(\cot\frac{1}{2}B-\tan\frac{1}{2}C)$.....(7).

Dividing (1) by (2) and (3) respectively, and clearing of fractions, we have

$b[x\cot\frac{1}{2}A+2\surd(xy)+y\cot\frac{1}{2}B]=c[x\cot\frac{1}{2}A+2\surd(xz)+z\cot\frac{1}{2}C]$ ....(8),

$a[x\cot\frac{1}{2}A+2\surd(xy)+y\cot\frac{1}{2}B]=c[y\cot\frac{1}{2}B+2\surd(yz)+z\cot\frac{1}{2}C]$ ....(9).

Subtracting x times (6) from (8), and y times (7) from (9), we have

$$b[x\tan\tfrac{1}{2}B+2\sqrt{(xy)}+y\cot\tfrac{1}{2}B]=c[x\tan\tfrac{1}{2}C+2\sqrt{(xz)}+z\cot\tfrac{1}{2}C]\ \ldots\ldots\ldots\ldots\ (10),$$

$$a[x\cot\tfrac{1}{2}A+2\sqrt{(xy)}+y\tan\tfrac{1}{2}A]=c[y\tan\tfrac{1}{2}C+2\sqrt{(yz)}+z\cot\tfrac{1}{2}C]\ \ldots\ldots\ldots\ldots\ (11).$$

Multiplying (10) by (4), and (11) by (5), and extracting the square root, we have

$$\sqrt{x}\sin\tfrac{1}{2}B+\sqrt{y}\cos\tfrac{1}{2}B=\sqrt{x}\sin\tfrac{1}{2}C+\sqrt{z}\cos\tfrac{1}{2}C\ \ldots\ldots\ldots\ldots\ (12),$$

$$\sqrt{x}\cos\tfrac{1}{2}A+\sqrt{y}\sin\tfrac{1}{2}A=\sqrt{y}\sin\tfrac{1}{2}C+\sqrt{z}\cos\tfrac{1}{2}C\ \ldots\ldots\ldots\ldots\ (13).$$

Subtracting (13) from (12), we find

$$\frac{\sqrt{y}}{\sqrt{x}}=\frac{\sin\frac{1}{2}C+\cos\frac{1}{2}A-\sin\frac{1}{2}B}{\sin\frac{1}{2}C-\sin\frac{1}{2}A+\cos\frac{1}{2}B}=\frac{\cos\frac{1}{2}C+\cos\frac{1}{2}(2B+C)}{\cos\frac{1}{2}C+\cos\frac{1}{2}(2A+C)}=\frac{\cos\frac{1}{2}B\cos\frac{1}{2}(\pi-A)}{\cos\frac{1}{2}A\cos\frac{1}{2}(\pi-B)}=\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}\ldots(14).$$

Similarly we find

$$\frac{\sqrt{z}}{\sqrt{x}}=\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}C}\ \ldots\ldots\ldots\ldots\ (15).$$

Substituting the value of $\sqrt{y}$ from (14) in (1), we have

$$\left[\cot\tfrac{1}{2}A+2\left(\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}\right)+\cot\tfrac{1}{2}B\left(\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}\right)^2\right]x=c,$$

or

$$\left[\frac{1-\tan^2\frac{1}{2}A}{2\tan\frac{1}{2}A}+2\left(\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}\right)+\left(\frac{1-\tan^2\frac{1}{2}B}{2\tan\frac{1}{2}B}\right)\left(\frac{1+\tan\frac{1}{2}A}{1+\tan\frac{1}{2}B}\right)^2\right]x=c,$$

whence

$$x=\frac{2c\tan\frac{1}{2}A\tan\frac{1}{2}B(1+\tan\frac{1}{2}B)}{(1+\tan\frac{1}{2}A)(\tan\frac{1}{2}A+\tan\frac{1}{2}B)(1-\tan\frac{1}{2}A\tan\frac{1}{2}B)[1+\tan\frac{1}{2}(A+B)]},$$

$$=\frac{c\sin\frac{1}{2}A\sin\frac{1}{2}B(1+\tan\frac{1}{2}B)}{\sin\frac{1}{2}(A+B)(1+\tan\frac{1}{2}A)[1+\tan\frac{1}{2}(\pi-C)]},\ =\frac{r(1+\tan\frac{1}{2}B)}{(1+\tan\frac{1}{2}A)[1+\tan\frac{1}{2}(\pi-C)]},$$

$$=\frac{\frac{1}{2}r(1+\tan\frac{1}{2}B)(1+\tan\frac{1}{2}C)}{1+\tan\frac{1}{2}A}.$$

Substituting the value of x in (14) and (15), we find

$$y=\frac{\frac{1}{2}r(1+\tan\frac{1}{2}A)(1+\tan\frac{1}{2}C)}{1+\tan\frac{1}{2}B},\quad\text{and}\quad z=\frac{\frac{1}{2}r(1+\tan\frac{1}{2}A)(1+\tan\frac{1}{2}B)}{1+\tan\frac{1}{2}C}.$$

An excellent solution was furnished by *William Hoover*, and a very brief one by Professor *Casey*. We hope to publish other solutions in a future No.

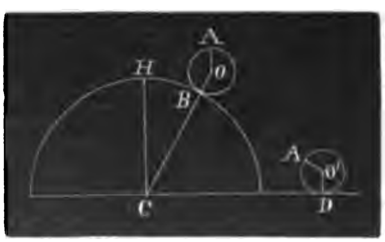
This celebrated problem has a history, and is known to mathematicians as "Malfatti's Problem."

187.—Proposed by Dr. JOEL E. HENDRICKS, M. A., Editor of the *Analyst*, Des Moines, Iowa.

The base of a hemisphere, whose radius is ten feet, rests on a horizontal plane, and a point A on the surface of a sphere whose radius is one foot is in contact with the vertex of the hemisphere.

If, from a slight disturbance of the equilibrium, the sphere is caused to *roll* off the hemisphere in the direction of the plane of a great circle through the points A and B on the surface of the sphere, and if the point B strikes the horizontal plane at a point D, distant CD from the center of the hemisphere; it is required to find the length of the line CD, and the relative position of the points A and B on the surface of the sphere.

Solution by WILLIAM HOOVER, Superintendent of Schools, Wapakoneta, Auglaize County, Ohio.



Put 1 foot $=a=$ the radius of the sphere, and 10 feet $=b=$ the radius of the hemisphere; let O be the position of the center of the sphere at the point of separation, v its orbital velocity, t the time in moving from O to O′ when the point B strikes the horizontal plane at D, H the vertex of the hemisphere, and ϕ the angle OCH.

We have (*Routh's Rigid Dynamics*, 2d Ed., pp. 103, 104)

$$\cos\phi=\tfrac{10}{17}\ldots(1),\ \text{and}\ v=\sqrt{[\tfrac{10}{7}g(a+b)(1-\cos\phi)]}\ldots(2).$$

We also find $\frac{v}{a}=$ angular velocity of the sphere, and $\frac{a+b}{a}\phi$ $=$ the angle through which the point A has revolved when the center of the sphere is at O. Also

$$vt\sin\phi+\tfrac{1}{2}gt^2=(a+b)\cos\phi-a\ \ldots\ldots\ (3),\quad\text{and}\quad CD=(a+b)\sin\phi+vt\cos\phi\ \ldots\ldots\ (4).$$

From (3), $$t=\frac{1}{g}\left(\pm\sqrt{[2g(a+b)\cos\phi-2ag+v^2\sin^2\phi]}-v\sin\phi\right)\ldots\ldots\ldots\ldots(5).$$

Substituting the value of v from (2) in (4) and (5), and the resulting value of t in (4), and reducing,

$$CD=(a+b)\sin\phi\pm\cos\phi\sqrt{\{\tfrac{20}{7}(a+b)(1-\cos\phi)[(a+b)\cos\phi+\tfrac{5}{7}(a+b)(1-\cos\phi)\sin^2\phi-a]\}}$$
$$-\tfrac{10}{7}(a+b)(1-\cos\phi)\sin\phi\cos\phi.$$

The angular distance described by the point A is

$$\frac{a+b}{a}\phi+\frac{vt}{a}=\frac{1}{a}\Big[(a+b)\phi\pm\sqrt{\{\tfrac{20}{7}(a+b)(1-\cos\phi)[(a+b)\cos\phi}$$
$$\overline{+\tfrac{5}{7}(a+b)(1-\cos\phi)\sin^2\phi-a]\}}\Big]-\frac{10\sin\phi}{7a}(a+b)(1-\cos\phi).$$

Restoring the numbers, CD $=11.646+$ feet, and $\frac{a+b}{a}\phi+\frac{vt}{a}=15.043+$, which divided by 2π gives 2.39416+, the number of revolutions of the sphere before impinging on the horizontal plane at D. Hence the angle AO′D $=2\pi\times0.39416=141^\circ\ 53'\ 51''$.

Solved also by the *Proposer*.

☞ The other solutions due in this No. are very reluctantly deferred till the publication of the next No.

List of Contributors to the Senior Department.

Solutions have been received from the following persons to the Problems indicated by the numbers:

E. B. Seitz, Member of the London Mathematical Society, Professor of Mathematics, North Missouri State Normal School, Kirksville, Missouri, 175, 176, 177, 178, 183, 184, 186, 190, 194, 196, 198, 201, 202, 204, 205, 206, 207, 209, 210, 215 and 216.

Walter Siverly, Oil City, Venango Co., Pa., 175, 176, 177, 182, 189, 190, 191, 192, 195, 196, 198, 199, 203, 206, 207, 211, 212, 213, 214 and 216.

William Hoover, Superintendent of Schools, Wapakoneta, Anglaize Co., Ohio, 175, 176, 177, 178, 179, 182, 183, 186, 187, 190, 191 and 203.

J. F. W. Scheffer, late Professor of Mathematics and German, Mercersburg College, Mercersburg, Franklin Co., Pa., 175, 176, 177, 178, 182, 195 and 196.

Charles H. Kummell, U. S. Coast and Geodetic Survey Office, Washington, D. C., 175, 176, 178, 180, 181 and 213.

DeVolson Wood, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, N. J., 175, 176, 183, 184, 188 and 192.

Edward A. Bowser, Professor of Mathematics and Engineering, Rutgers College, New Brunswick, New Jersey, 183, 195, 207, 211 and 216.

F. P. Matz, M. A., late Professor of Mathematics, Military and Scientific School, King's Mountain, North Carolina, 180, 190, 199, 201 and 212.

George Eastwood, Saxonville, Middlesex Co., Massachusetts, 175, 176, 177, 178 and 212.
L. G. Barbour, Professor of Mathematics, Central University, Richmond, Kentucky, 203, 212, 214 and 216.
W. E. Heal, Marion, Grant Co., Indiana, 177, 182, 191 and 195.
Dr. David S. Hart, M. A., Stonington, New London Co., Conn., 177, 191 and 197.
Professor W. P. Casey, C. E., San Francisco, California, 176, 178 and 186.
E. J. Edmunds, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana, 176, 178 and 186.
H. T. J. Ludwig, Professor of Mathematics, North Carolina College, Mount Pleasant, Cabarrus Co., N. C., 182 and 190.
Lucius Brown, Hudson, Middlesex Co., Mass., 193 and 210.
L. P. Shidy, U. S. Coast and Geodetic Survey Office, Washington, D. C., 177 and 191.
Sylvester Robins, North Branch Depot, Somerset Co., N. J., 177 and 179.
K. S. Putnam, Rome, Oneida Co., N. Y., 190 and 191.
C. A. O. Rosell, B. A., Teacher of Mathematics at the Carroll Institute, Reading, Pa., 175 and 176.
Hon. Josiah H. Drummond, LL. D., Portland, Maine, 177 and 191.
Samuel Roberts, M. A., F. R. S., President of the London Mathematical Society, London, England, 185 and 193.
Reuben Davis, Bradford, Stark County, Illinois. 191 and 197.
Julian A. Pollard, Goshen, Orange Co., N. Y., 175 and 177.
W. L. Harvey, Maxfield, Penobscot Co., Maine, 175 and 179.
Rev. W. J. Wright, M. A., Ph. D., Burlington, Chittenden Co., Vermont, 208.
Professor Ormond Stone, M. A., Astronomer at the Cincinnati Observatory, Mt. Lookout, Hamilton Co., Ohio, 216.
Dr. S. H. Wright, M. A., Ph. D., late Mathematical Editor of the *Yates County Chronicle*, Penn Yan, N. Y., 184.
Professor Hugh S. Banks, Instructor in English and Classical Literature, Newburg, N. Y., 189.
Marcus Baker, U. S. Coast and Geodetic Survey Office, Washington, D. C., 186.
Henry Heaton, B. S., Perry, Dallas Co., Iowa, 198.
Rev. U. Jesse Knisely, Ph. D., Newcomerstown, Tuscarawas Co., Ohio, 186.
George H. Harvill, Colfax, Grant Co., Louisiana, 212.
C. C. Robins, Princeton, New Jersey, 177.
D. J. McAdam, M. A., Professor of Mathematics, Washington and Jefferson College, Washington. Pa., 214.
William Woolsey Johnson, Member of the London Mathematical Society, Professor of Mathematics St. John's College, Annapolis, Maryland, 193.
J. M. Arnold, South Boston, Massachusetts, 216.
Alex. S. Christie, U. S. Coast and Geodetic Survey Office, Washington, D. C., 202.
Isaac H. Turrell, Principal 4th District School, Cincinnati, Ohio, 186.
Dr. Joel E. Hendricks, M. A., Editor and Publisher of the *Analyst*, Des Moines, Iowa, 187.

The first prize for solution of Problem 216 is awarded to Professor E. B. Seitz, Kirksville, Mo., and the second to J. M. Arnold, South Boston, Mass.

No solution of Problem 217 has been received.

PROBLEMS.

243.—Proposed by REUBEN DAVIS, Bradford, Stark County, Illinois.

It is required to find three positive integral numbers, such that their sum is a cube, and also the sum of any two of them a cube.

Reproposed to correct errors; as stated in No. 5, it is impossible.

276.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

If $\frac{p_n}{q_n}$ be the last convergent in the first period of $\sqrt{A}$ expanded as a continued fraction, and r the integral part of $\sqrt{A}$, show that $$p_n = rq_n + q_{n-1}.$$

277.—Proposed by SAMUEL G. CAGWIN, New London, Oneida County, N. Y.

A given volume v of metal is to be made into a rectangular vessel of uniform thickness a. Determine the dimensions of the vessel so that its capacity shall be a maximum, (1) when it has no cover, and (2) when it has a cover.

278.—Proposed by E. J. EDMUNDS, B. S., Professor of Mathematics, Southern University, New Orleans, Louisiana.

When two parabolas having a common tangent and a common focus cut each other under a constant angle, find the locus of their point of intersection.

279.—Proposed by Hon. JOSIAH H. DRUMMOND, LL. D., Portland, Maine.

Find three positive whole numbers in arithmetical progression whose sum shall be a square, and such that if p^4 be added to each the several sums shall be squares; the numbers varying when different values are given to p, and p being any number.

280.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

If the feet of the perpendiculars of a plane triangle be joined two and two a new triangle is formed, which call the *first pedal triangle;* if in like manner the feet of the perpendiculars of the first pedal triangle be joined another triangle is formed, which call the *second pedal triangle;* and so on. It is required to find expressions for the sides and angles of the nth *pedal triangle* in terms of the functions of the original triangle.

281.—Proposed by SYLVESTER ROBINS, North Branch Depot, Somerset County, New Jersey.

Investigate that series of parallelopipeds in which the law requiring each edge and solid diagonal of the nth term to be expressed by n figures holds to the greatest possible number of terms.

282.—Proposed by Professor W. P. CASEY, C. E., San Francisco, California.

If the lines AO, BO, CO from the angles of a triangle ABC to the center O of the inscribed circle be produced to meet the opposite sides in α, β, γ, and from these points as centers with the respective distances αO, βO, γO as radii three circles be described the sum of the reciprocals of the radii of all the circles respectively touching these circles is equal to four times the reciprocal of the radius of the inscribed circle.

283.—Proposed by Professor DAVID TROWBRIDGE, M. A., Waterburg, Tompkins County, N. Y.

If $$(1-2pc+c^2)^{-m} = 1 + cP_1 + c^2P_2 + \ldots\ldots c^iP_i + \ldots\ldots,$$
m being integral or fractional, and P_i a function of p, prove that
$$\int_{-1}^{+1} P_iP_n(1-p^2)^{m-\frac{1}{2}}dp = 0,$$
i and n being different integers.

284.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society Erie, Erie County, Pa.

Two equal circles, radii r, intersect, the center of each being on the circumference of the other. A circle is drawn touching that diameter of the right-hand circle which joins the centers of the given circles, and the circumferences of both circles, the right-hand one internally and the other externally; a circle is drawn touching the one last drawn and the circumferences of both the given circles; and so on.

Find the radius of the nth circle.

285.—Proposed by Dr. DAVID S. HART, M. A., Stonington, New London County, Connecticut.

Find thirty-four biquadrate numbers whose sum shall also be a biquadrate number.

286.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Find the average distance between two points taken at random in the surface of a given ellipse.

287.—Proposed by CHARLES GILPIN, Jr., Philadelphia, Pennsylvania.

A cup of wine is suspended over a cup of equal capacity full of water; through a small hole in the bottom the wine drips into the water cup, and the mixture drips out at the same rate. When the wine cup is empty, what part of the contents of the lower cup is water?

288.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Three points being taken at random in the surface of a circle, find the chance that the triangle formed by joining them is acute.

289.—Proposed by Miss CHRISTINE LADD, B. A., Fellow of Johns Hopkins University, Baltimore, Maryland.

If R is the radius of the circumscribed circle of a triangle ABC, r that of the inscribed circle, ρ that of the circle inscribed in the orthocentric triangle, I the center of the inscribed circle of the triangle ABC, and Q that of the circle inscribed in the triangle formed by joining the middle points of the sides of ABC, show that $$(QI)^2 = \tfrac{1}{4}(7R\rho - 6Rr + 3r^2 + 2R^2).$$

290.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

A circle, radius r, touches another circle, radius R, internally. Find the average area of all the circles that can be drawn touching the two given circles.

291.—Proposed by J. C. GLASHAN, Ottawa, Ontario, Canada.

From one solution of the equation $x^2 + y^2 - z^2 = a$ an infinity of others may be derived thus: Write the given values in order x, z, y in a vertical column. Make a second column thus: take the sum of the two upper numbers for a new top number; the sum of the two lower numbers for a new bottom number, and the sum of all three for a new middle number. Repeat the operation on this second column so as to get a third; the numbers in this third column will give a new solution of the proposed equation. Continue the process to any number of columns, the odd columns will give new solutions. The sign of any number in an odd column may be taken either positive or negative.

The following is an example of the process applied to $x^2 + y^2 - z^2 = 10$:

I		II		III		IV	
5	13	33	81	197	477	755	
8	20	48	116	280	287	836	etc.
7	15	35	83	—199	81	359	

(Adapted from Quest. 4102, by T. T. Wilkinson, in the *Educational Times Mathematical Reprint*, Vol. XX, p. 20.)

Generalize the above so as to obtain from one solution of $x_1^2 + x_2^2 + x_3^2 + \ldots\ldots + x_n^2 - y^2 = a$ an infinity of solutions.

Example.—From $0^2+0^2+1^2-1^2=0$ obtain $2^2+2^2+1^2-3^2=0$ and from this $2^2+3^2+6^2-7^2=0$.

292.—Proposed by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics at the North Missouri State Normal School, Kirksville, Adair County, Missouri.

Two points are taken at random within a circle on opposite sides of a given diameter, and a third point is taken anywhere within the circle; find the average area of the triangle formed by joining the three points.

293.—Proposed by GEORGE EASTWOOD, Saxonville, Middlesex County, Massachusetts.

On what days of *any* year in a given latitude will the contour of the shadow swept out by a vertical pole, on a horizontal plane, have the greatest length?

294.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

A dog seeing a fox at the center of a circular field leaped over the fence and "went for him," the fox running in a straight line and the dog directly towards him.

Find the chance that the fox will escape from the field.

295.—Proposed by Professor ASHER B. EVANS, M. A., Principal Lockport Union School, Lockport, Niagara Co., N. Y.

Four circles of given unequal radii r_1, r_2, r_3, r_4 have their centers at the vertices of a quadrilateral ABCD whose angles are variable. If $AB = r_1 + r_2$, $BC = r_2 + r_3$, $CD = r_3 + r_4$, $DA = r_4 + r_1$, what relation must exist among the angles A, B, C, D that that part of ABCD exterior to the four sectors may be a maximum?

296.—Proposed by E. B. SEITZ, M. L. M. S., Professor of Mathematics, North Missouri State Normal School, Kirksville, Mo.

Three points M, N, P are taken at random within a triangle ABC, and lines AMD, BNE, CPF are drawn to meet the sides of ABC in D, E, F, intersecting each other in R, S, T.

Find the average area of the triangle RST.

297.—Proposed by SAMUEL ROBERTS, M. A., F. R. S., President of the London Mathematical Society, London, England.

Three circles A, B, C are inscribed or escribed about the same triangle; show that in the triangle formed by the three double tangents $\overline{AB}$ touching A and B, $\overline{BC}$ touching B and C and $\overline{CA}$ touching C and A, an infinite number of triangles can be inscribed such that the sides opposite the vertices on $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ touch the circles C, A, B respectively.

298.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie County, Pa.

There is a cylindrical tower, radius a, at the center of a circular park, radius R. A hunter on the outside of the fence saw a rabbit in the park which "scud" around the tower out of sight and "made for the fence." The hunter ran around by the fence, but the rabbit kept "*just* out of sight" all the time.

Required the equation to the *curve* the rabbit described, and the distance it ran to reach the fence.

299.—Proposed by DEVOLSON WOOD, M. A., C. E., Professor of Mathematics and Mechanics, Stevens Institute of Technology, Hoboken, New Jersey.

If each of n vessels closely connected in a circuit contains a different liquid, each q gallons, and the liquids circulate by flowing uniformly in one direction at the rate of a gallons per minute, mixing uniformly, how much of each liquid will there be in any one of the vessels at the end of time t?

300.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

During a heavy rain storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond?

301.—Proposed by E. B. SEITZ, Member of the London Mathematical Society, Professor of Mathematics at the North Missouri State Normal School, Kirksville, Adair County, Missouri.

Let M, N, P, Q, R, S be six random points within a circle; find (1) the chance that each of the chords through M and N, P and Q, R and S, intersects the other two; and (2) the chance that each of the chords intersects the other two, and the triangle formed contains the center of the circle.

302.—Proposed by ARTEMAS MARTIN, M. A., Member of the London Mathematical Society, Erie, Erie Co., Pa.

Having given the polar equation $r = f(\theta)$ of a curve, deduce therefrom by a general method, without changing to rectangular co-ordinates, its *intrinsic* equation; and by it find the intrinsic equation of the spiral of Archimedes, and of its involute.

Solutions of these problems should be received by March 1, 1882.

EDITORIAL NOTES.

We deeply regret that it is our sad duty to record the death of the eminent mathematician Professor BENJAMIN PEIRCE, LL. D., F. R. S., of Harvard University, which occurred October 6, 1880.

The following notice is extracted from a newspaper cutting kindly sent us by Mr. S. C. GOULD of Manchester, New Hampshire:

"He was born in Salem in 1809, and from his earliest years manifested a taste for mathematical studies, a taste which was developed by the formerly well-known teacher, Dr. Bowditch, and which led him, while at college, into complete absorption in his favorite pursuit. It is related of him that, during the last year of his course, he neglected all his other studies and attended no recitations whatever, devoting fourteen hours a day to mathematics. After graduating, he continued his mathematical studies at the same tension, and in 1842 he became Perkin's professor of astronomy and mathematics at Harvard college. In 1867 he succeeded Prof. Bache as superintendent of the United States coast survey, resigning the latter position in March, 1874. As a mathematician he attained the first rank, and he had few, if any, compeers in his highest intellectual labors. He was noted for his directness and conciseness of demonstration, and by the intuitive insight with which he approached the most difficult problems. His published works are few, although his contributions to the science of mathematics are most important, and his textbooks and elementary treatises are widely circulated. So abstruse are some of his works that, as he himself said, only one man besides himself has been able to understand them. To Prof. Peirce belongs the distinction of being one of the founders of a new branch of mathematics, the final form of which is not yet determined, but which may prove to be the great event in the mathematical history of this country—'Linear Associative Algebra.' This book has been published in an edition of some fifty copies. As an astronomer Prof. Peirce's record is high, although he has written no work on the subject."

Our thanks are due Prof. E. B. SEITZ for valuable assistance in correcting the proof sheets of this No.

Prof. CASEY, Prof. WOOD, Prof. BEMAN, J. S. ROYER, SYLVESTER ROBINS, and many others, are entitled to thanks for their success in procuring subscribers.

Prof. W. W. BEMAN of Michigan University, Ann Arbor, has been elected a Member of the London Mathematical Society.

Miss CHRISTINE LADD, B. A., has been elected a Fellow of Johns Hopkins University.

Prof. EDWARD BROOKS, Ph. D., Principal of the Pennsylvania State Normal School, Millersville, Pa., has returned from abroad and is again hard at work among his classes.

Prof. E. J. EDMUNDS, B. S., has returned from France, and is now Professor of Mathematics in the Southern University, New Orleans, Louisiana.

Prof. F. P. MATZ, M. A., late of King's Mountain Military and Scientific School, is a student at the Johns Hopkins University.

The Mathematical Department in the *Yates County Chronicle*, so ably conducted by Dr. S. H. WRIGHT, M. A., Ph. D., for more than eight years, has been very abruptly discontinued. The Dr. did not even have an opportunity of saying a parting word to his contributors. It was begun February 29, 1872, and ended October 22, 1880. The 2500 problems and their solutions published would fill three or four large octavo volumes and constitute a most valuable collection.

The Mathematical Department in the *Wittenberger* has also been discontinued. The October Number, 1880, contains Mr. HOOVER's valedictory.

The type-setting and printing of this No. of the VISITOR has been all done by the Editor, and he deeply regrets that the delays occasioned by the breaking of his press and various other causes, and having to move, compel him to "cut it short."

The present No. closes the first volume of the VISITOR. In order that the Editor may get the rest he so much needs and *must* have, no VISITOR will be issued in July this year. The next No., it is hoped, will be ready in January, 1882. All who have paid for another No. will receive No. 1 of Vol. II.

A limited number of bound copies of Vol. I can be supplied at $3.50.

As some of our contributors may desire to procure copies of their solutions published in the VISITOR, we will state here that we can supply them hereafter at $1.50 per hundred, printed on good paper the size of a VISITOR leaf, if the solution does not occupy more than one page; if more than one page, $1.00 per page for 100 copies.

The Editor having had so many applications for his photograph has decided to have his portrait engraved. It will be printed on heavy paper, the size of a VISITOR leaf, suitable either for framing or binding with the VISITOR, and supplied to subscribers at 25 cents. We will be pleased to receive orders for the portrait at once so that we can determine the number to be printed. Say whether you want it bound with the VISITOR or not—or order *two*, one for framing and the other to be bound in the VISITOR, and the two will be furnished for 40 cents.

Contributors are requested to send in their solutions of the problems proposed in the Junior Department of No. 5 as soon as practicable.

We must ask our subscribers not to send us their individual checks nor drafts on private or country banks, as we are subjected to a "shave" on them. Postal Money-Orders and Drafts on New York preferred. Our Canada subscribers will please not send Canada postage stamps—we can not use them. English postage stamps are always acceptable: they can be used in making remittances to Londen.

NOTICES OF BOOKS AND PERIODICALS.

An Elementary Treatise on the Differential and Integral Calculus. With Numerous Examples. By Edward A. Bowser, Professor of Mathematics and Engineering in Rutgers College. 12mo, pp. 395. New York: D. Van Nostrand, 23 Murray street.

An elementary work designed as a text-book for colleges and scientific schools, wel. supplied with appropriate examples, some of them selected from the VISITOR.

A *differential* is defined as the *difference between two consecutive values* of a variable or function; and *consecutive values* of a variable or function, as values which differ from each other by *less than any assignable quantity*.

It is a work of rare excellence, beautifully printed on fine paper, and we can heartily recommend it to those in search of a clear exposition of the principles of this (to many minds) seemingly mysterious subject.

Prof. Bowser's Mathematical Works (Analytical Geometry, and Treatise on the Calculus,) have been adopted already as the text-books in the following Institutions: University of Penna., Philadelphia, Pa.; Free Institute, Worcester, Mass.; Wesleyan University, Middletown, Conn.; University of Georgia, Athens, Ga.; Yale College, New Haven, Conn.; Rutgers College, New Brunswick, N. J.; North Carolina College, Mt. Pleasant, N. C.; Madison University, Hamilton, N. Y.; Polytechnic Institute, Brooklyn, N. Y.; Racine College, Racine, Wis.; University of N. C.; Chapel Hill, N. C.; State College, Austin, Texas; Syracuse University, Syracuse, N. Y.; Hanover College, Hanover, Ind.; Rensellaer Polytechnic Institute, Troy, N. Y.; Washington University, St. Louis, Mo.

For table of contents, and price, see advertisement on second page of the cover of this No.

A Synopsis of Elementary Results in Pure and Applied Mathematics: Containing Propositions, Formulæ, and Methods of Analysis, with Abridged Demonstrations. By G. S. Carr, B. A., late Prizeman and Scholar of Gonville and Caius College, Cambridge. Vol. I, Part I. 8vo, pp. 280. London: C. F. Hodgson and Son.

Part I of Vol. I contains seven sections: Mathematical Tables; Algebra; Theory of Equations and Determinants; Plane Trigonometry; Spherical Trigonometry; Elementary Geometry; Geometrical Conics. Part II of Vol. I, which is in the Press, will contain the following sections: Differential Calculus; Integral Calculus; Calculus of Variations; Differential Equations; Plane Co-ordinate Geometry; Solid Co-ordinate Geometry. Vol. II is in preparation and will be devoted to Applied Mathematics and other Branches of Pure Mathematics.

Chiefly valuable as a work of reference; it is well printed, and profusely illustrated with excellent diagrams.

An Elementary Treatise on Trigonometry, with Numerous Examples and Applications. Designed for the use of High Schools and Colleges. By J. Morrison, M. D., M. A., Principal of the Walkerton High School. 12mo, pp. 332. Toronto: Canada Publishing Co. (Limited).

A very full and complete treatise on Plane Trigonometry, containing great store of interesting and well-chosen examples, and many elegant solutions.

The Teacher's Hand-Book of Algebra; Containing Methods, Solutions and Exercises Illustrating the Latest and Best Treatment of the Elements of Algebra. By J. A. McLellan, M. A., LL. D., High School Inspector for Ontario. 12mo, pp. 229.

Key to the Teacher's Hand-Book of Algebra. By the same Author. Second Edition—Revised. 12mo, pp. 199. Toronto: W. J. Gage and Company.

The "Hand-Book" is really a book of examples and problems in Algebra—and a very good one, too. It contains, however, a large number of solutions, and gives complete explanations and illustrations of important topics. A valuable little work for both teacher and student. In the "Key" full solutions are given of all the difficult problems.

We regret that we can not praise the printing of these excellent books. Poor printing is a lamentable fault we have noticed in all the Canada books that we have seen.

Ray's New Higher Arithmetic. A Revised Edition of the Higher Arithmetic. By Joseph Ray, M. D., Late Professor in Woodward College. 12mo, pp. 408. Cincinnati & New York: Van Antwerp, Bragg & Co.

The work of revision has been very efficiently done by Prof. J. M. Greenwood, M. A., Superintendent of Public Schools, Kansas City, Mo., and Rev. U. Jesse Knisely, Ph. D., of Newcomerstown, O., and this is now one of the best Higher Arithmetics "in the field."

Mathematical Questions, with their Solutions, from the "Educational Times," with many Papers and Solutions not published in the "Educational Times." Edited by W. J. C. Miller, B. A., Registrar of the General Medical Council. Vols. XXXIII and XXXIV. From January to July 1880, and from July to December, 1880. 8vo, boards, pp. 116 and 120. London: C. F. Hodgson & Son.

Vol. XXXIII contains 5 papers and solutions of 138 problems. Vol. XXXIV contains 1 paper and solutions of 147 problems. Many of the finest "Average" and "Probability" solutions are by our valued contributor Professor Seitz.

The *Educational Times* is published monthly, and contains a most valuable Mathematical Department. The *Reprint* is issued in half-yearly volumes.

The Editor of the Visitor will be pleased to furnish the *Times* at $2.00 a year, and the *Reprint* at $1.75 per Vol.

The Analyst: A Journal of Pure and Applied Mathematics. Edited and Published by J. E. Hendricks, M. A., Des Moines, Iowa. Bi-monthly; 8vo, pp. 32. $2.00 a year. Vol. VIII.

As interesting and valuable as ever, and deserves the continued support of mathematicians.

The School Visitor. Devoted to the Study of Mathematics and English Grammar. Edited by John S. Royer and Thomas Ewbank. Monthly; pp. 16. $0.75 a year; single numbers, 10 cents. Ansonia, O.: Published by John S. Royer.

Has grown some, and each No. this year will have a nice cover. The Mathematical Department contains well-selected and interesting Problems and Solutions in the Elementary Branches, and is illustrated with beautiful diagrams. Its merits deserve to be crowned with success—every teacher should have a copy.

Educational Notes and Queries. Issued monthly, except in the vacation months, July and August. Edited and Published by Prof. W. D. Henkle, Salem, Ohio. $1.00 a year.

Contains a vast fund of curious and valuable information, besides Mathematical Problems and Solutions.

The Canada School Journal. Monthly. Toronto, Canada: W. J. Gage & Co. $1.00 a year in advance.

An excellent educational journal. The Mathematical Department is under the able management of Alfred Baker, M. A.

Barnes' Educational Monthly. New York: A. S. Barnes & Co. $1.50 a year.

A live periodical, containing an interesting Mathematical Department.

The Canada Educational Monthly and School Chronicle. Toronto: Canada Educational Monthly Publishing Co. $1.50 a year.

A very readable journal. The Arts Department, edited by Archibald MacMurchy, M. A., is devoted to Problems and Solutions, and Examination Papers.

Gage's School Examiner and Monthly Review. Vol. I. Toronto: W. J. Gage & Co. $1.00 a year.

A new candidate for public favor, "containing full sets of model Examination Papers, and a survey of current thought and art, with special reference to the teacher's point of view."

The Pennsylvania School Journal. Monthly. J. P. Wickersham, Editor. Lancaster: J. P. Wickersham & Co. $1.60 a year.

One of the best educational journals in the country.

New-England Journal of Education. Thomas W. Bicknell, Editor. Boston: 16 Hawley street. $3.00 a year; $2.50 in advance.

Issued weekly, contains a vast fund of educational intelligence, and should be read by all teachers.

The California Architect and Building Review. James E. Wolfe, Editor and Manager. Geo. H. Wolfe, Business Manager. San Francisco: San Francisco Architectural Publishing Co. Monthly. $2.00 per annum in advance. Single copies, 20 cents.

An excellent periodical devoted to the interests of the architects and builders of the Pacific Slope. In "Everybody's Column" questions of intererest are discussed, and problems solved.

It contains also much valuable information of interest to the general reader, relating to house-drainage, sinks, &c. These matters of vital importance can not receive too much attention, and their neglect is almost sure to result in a visit from Death.

Discussion of a Geometrical Problem: With Bibliographical Notes. By Marcus Baker, U. S. Coast Survey, Washington, D. C. Extracted from the Bulletin of the Philosophical Society of Washington. Philadelphia: Collins, Printer, 705 Jayne St.

The problem discussed is the following:

"In a right-angled triangle there are given the bisectors of the acute angles: required to determine the triangle." Six solutions are given, and two constructions. This problem was proposed in the Ladies' Diary for 1797, and has appeared in many places since that time.

Micrometrical Measurements of 1054 *Double Stars*, observed with the 11-inch Refractor from January 1, 1878, to September 1, 1879, under the superintendence of Ormond Stone, M. A., Director. Cincinnati: Published by authority of the Board of Directors of the University.

On the Extra-Meridian Determination of Time, by means of a Portable Transit-Instrument. By Ormond Stone, M. A., Astronomer of the Cincinnati Observatory.

Three Approximate Solutions of Kepler's Problem. By H. A. Howe, M. A., Assistant at the Cincinnati Observatory.

CORRIGENDA.

No. 5.

Page 141, Problem 141, last line, first "any" should be "the"; line 1 of solution, *for* "constant" *read* counted; line 12, after "number" insert ν.

Page 147, lines 3 and 4, *for* "$\sqrt{(px-x^2)}$" *read* $\sqrt{(px-y^2)}$.

Page 149, line 14, in the denominator *for* "$(n+n)$" *read* $(n+1)$.

Page 155, line 1, after "outline" insert "in the expansion of $(b\pm c\pm d\pm e\pm x)^5$" and omit the sentence in ( ).

Page 158, Problem 237, the latitude should be given $=40°\ 10'$ N.

No. 6.

Page 178, solution of Problem 176, line 6, *for*

"$A=\frac{1}{3}\sqrt{[\frac{1}{2}(8q-4mp+2m^2n-m^4)]}$" *read* $A=\frac{1}{4}\sqrt{(16q-8mp+4m^2n-m^4)}$;

last line, *for* "$R=\frac{1}{2}\sqrt{\left(\frac{p^2+m^2q-4nq}{8q-4mp+2m^2n-m^4}\right)}$" *read* $R=\sqrt{\left(\frac{p^2+m^2q-4nq}{16q-8mp+4m^2m-m^4}\right)}$.

END OF VOL. I.